THE UNIVERSITY OF ZAMBIA
SCHOOL OF NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS & STATISTICS
MAT 2110–Engineering Mathematics I

Tutorial Sheet 2 **April 2024**

---

1. Find the first derivative of each of the following:

   (a) $y = e^{4\tanh\left(\sqrt{x}\right)}$ **(b)** $f(x) = \sin^{-1}(x^3 + x) + 5^x$ **(c)** $\ln(x + y) = xy - y^3$

   (d) $y = \cosh(\sin -1(x^2 \ln x))$ **(e)** $\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\ln(x^2 + y^2)$ **(f)** $\log_3 \sqrt{y} = x^2 - 1$

2. Find the n^{th} derivative of each of the following and use it to evaluate $y^{(9)}$ and $y^{(12)}$:

   (a) $y = \sqrt{5x + 1}$ **(b)** $y = \dfrac{1}{1 - 2x}$ **(c)** $y = \frac{1}{x} + \cos(2x)$

3. A ladder 12 metres long leans against a wall. The foot of the ladder is pulled away from the wall at the rate of $\frac{1}{2}$ metres per minute. At what rate is the top of the ladder falling when the foot of the ladder is 4 metres from the wall?
4. A spherical iron ball 10 centimetres in radius is coated with a layer of ice of uniform thickness that melts at a rate of 50 cubic centimetres per minute. Find the rate at which the thickness of ice is decreasing when the thickness is 15 centimetres.
5. Sand is pouring out of a tube at 1 cubic metres per second. It forms a pile which has the shape of a cone. The height of the cone is equal to the radius of the circle at its base. How fast is the sandpile rising when it is 2 metres high?
6. Find the maximum of the function

$$f(x) = (x + 1)^{\frac{1}{3}} - (x - 1)^{\frac{1}{3}} \quad \text{on} \quad [0, 1].$$

7. Consider the function

$$f(x) = x^{\frac{2}{3}}\left(\frac{5}{2} - x\right).$$

   (a) Determine the behaviour of f as $x \to \pm\infty$.

   (b) Determine the intervals where f is increasing and decreasing.

   (c) Locate the stationary points of f and determine their nature.

   (d) Hence, or otherwise, sketch the graph of f.

8. A window is being built and the top is a semicircle and the bottom is a rectangle. You have been given 15 metres of framing material. Let x be the radius of the semicircle.

   (a) Express the total area of the window in terms of x.

   (b) Find the dimensions of the window that will maximize its total area.

9. A box whose base length is 3 times the base width is to be constructed. The material used to build the top and the bottom cost zmw $10/cm^2$ and the material used to build the sides cost zmw $6/cm^2$. If the box must have a volume of 50 cm^3, determine the dimensions that will minimise the cost to build the box.
10. A rectangular soccer field with the maximum possible area is to be designed inside an elliptical sport field given by the graph of $\frac{x^2}{25} + \frac{y^2}{9} = 1$. The length and width of the rectangular soccer field are $2x$ and $2y$, respectively.

(a) Express the area of the rectangular soccer field in terms of x.

(b) Find the dimensions of the rectangular soccer field that will give the maximum area.

11. Find the equation of the tangent and normal to the following curves at the indicated point:

(a) $f(x) = x + \frac{2}{x}$ at $x = 1$ **(b)** $\frac{y^2}{4} - \frac{(x-1)^2}{5} = 1$ at $\left(3, -\frac{6}{\sqrt{5}}\right)$

12. Given the curve $y = (x-1)^3$ and the line $L : 3y + x = 0$, find the equation of all lines that are tangent to the curve and are also perpendicular to L.
13. Find the equation of the tangent to the curve $f(x) = x + \frac{4}{x^2}$ that is parallel to the $x-$axis.
14. Find all the points on the curve

$$y = x^2 + \sqrt{1 - x^2}$$

at which the tangent is perpendicular to the $x-$axis.

15. Use linearisation to approximate the value

(a) $\sqrt{80}$ **(b)** $(63)^{\frac{2}{3}}$ **(b)** $(124)^{\frac{2}{5}} - 1$

16. Find the linearisation of

$$e^y + y(x-2) = x^2 - 8$$

at $(3, 0)$ and use it to estimate y when $x = 2.98$.

17. Check that the hypotheses of the Rolle's Theorem for the following functions on the given intervals are satisfied. Then, find the value(s) c such that $f'(c) = 0$.
(a) $f(x) = \frac{1}{2}x - \sqrt{x}$ on $[0, 4]$
(b) $f(x) = \sin x - \cos x$ on $[\frac{\pi}{4}, \frac{9\pi}{4}]$
(c) $f(x) = 2^{2x+1} - 3.2^x + 1$ on $[-1, 0]$
(d) $f(x) = x^{\frac{2}{3}} - x^{\frac{1}{3}}$ on $[0, 1]$.
18. Verify the Mean Value Theorem for the following functions on the given interval $[a, b]$. Then, find the value(s) $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(a) $f(x) = \dfrac{x-4}{x-3}$, $[4, 6]$

(b) $f(x) = \sqrt{x-1}$, $[1, 3]$

(c) $f(x) = \csc x$, $[\frac{\pi}{6}, \frac{\pi}{2}]$

19. Evaluate each of the following limits:

(a) $\lim_{x \to 1} \left[\dfrac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}\right]$, when $n \geq 2$

(b) $\lim_{x \to 0} \left[\dfrac{x - \sin x}{x - \arctan x}\right]$

(c) $\lim_{x \to 0^+} \left[(\sin x)^{\tan x}\right]$

(d) $\lim_{x \to \infty} \left[\left(\dfrac{\ln x}{x}\right)^{\frac{1}{\ln x}}\right]$

(e) $\lim_{x \to \infty} x^2 e^{-x}$

(f) $\lim_{x \to 0} \left[\left(\dfrac{1 - \cos x}{x}\right)^x\right]$