KINEMATICS OF PARTICLES

2.1 Introduction to kinematics of particles

- **We start our study of kinematics by first discussing the motions of points or particles**

- **There are a number of ways in which a particle can be described**

- **The choice of the most convenient or appropriate method depends largely on experience**

Introduction to Kinematics of particles continued ..

- **To get an overview of several methods developed in this lesson refer to the figure below**. P is a particle moving along some general path.

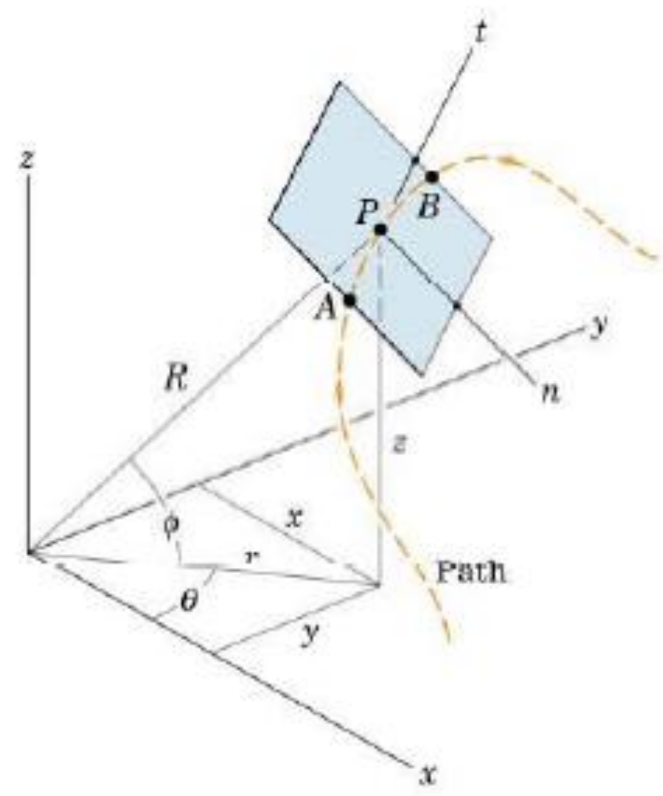


Figure 2/1

Introduction to Kinematics of particles continued ..

- **The position of particle P at any time t can be described by** specifying:

- its rectangular coordinates x, y, z

- Its cylindrical coordinates r, θ, z

- Or its spherical coordinates R, θ, φ

- It may also be described by measurements along the tangent *t* and the normal *n* to the curve

Introduction to kinematic of particles continued..

- When there are physical guides to the motion, the motion is said to be **constrained**

- Without physical guides the motion is **unconstrained**

- A small rock tied to the end of a string and whirled in a circle undergoes constrained motion

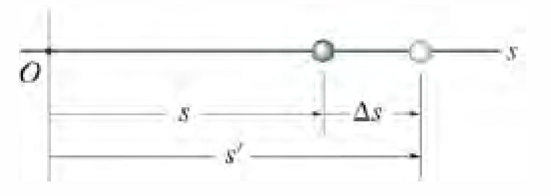
- Motion described by coordinates measured from a fixed reference frame is said to be absolute motion analysis

- Motion described by coordinates measured from moving frame is said to be relative motion analysis

PLANE MOTION

- With the conceptual picture from figure 2/1 we now restrict our attention in the first part to **PLANE MOTION**
- Here all movement occur in a single plane
- We begin with rectilinear motion

2.2 RECTILINEAR MOTION.



- Consider a particle along a straight line

  in the limit $v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$, $v = \frac{ds}{dt} = \dot{s}$

- Average acceleration in the limit $a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$

$$a = \frac{dv}{dt} = \dot{v} \quad \text{or} \quad a = \frac{d^2 s}{dt^2} = \ddot{s}$$

- Removing dt from the above equations results in

$$v\,dv = a\,ds \quad \text{or} \quad \dot{s}\,d\dot{s} = \ddot{s}\,ds$$

KINEMATICS – Plane Motion: Rectilinear Motion

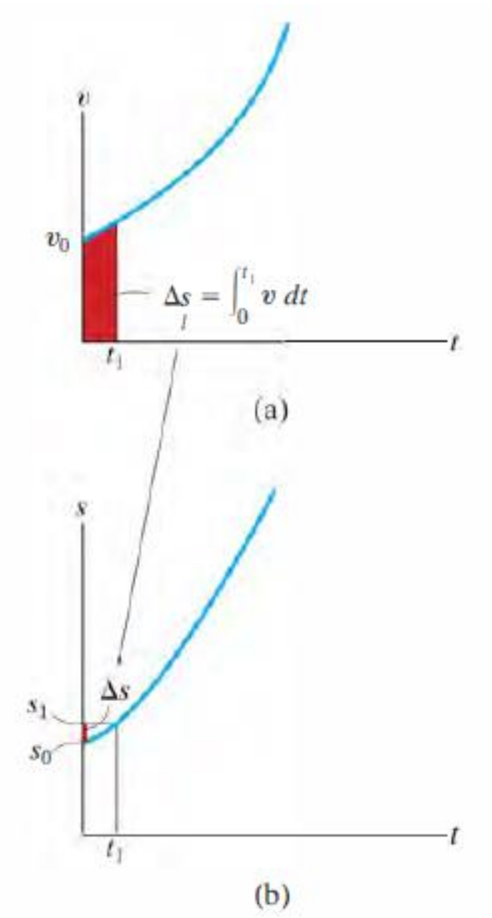
- **Rectilinear motion continued …**
- **Area under the v-t curve**

$$\int_{s_1}^{s_2} ds = \int_{t_1}^{t_2} v dt$$

$$s_2 - s_1 = (area\ under\ v - t\ curve$$

$$\Delta s = \int v\, dt$$

$$\text{displacement} = \begin{array}{c}\text{area under} \\ v\text{–}t \text{ graph}\end{array}$$



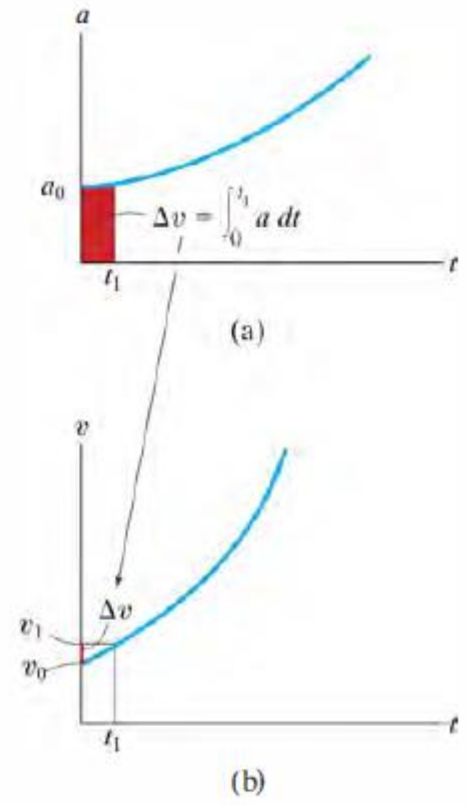
- **Rectilinear motion continued …**
- **Area under the a-t curve**

$$\int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} a\,dt$$

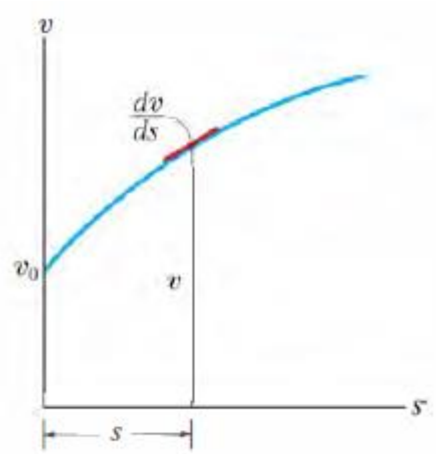
$$v_2 - v_1 = (area\ under\ a - t\ curve$$

$$\Delta v = \int a\,dt$$

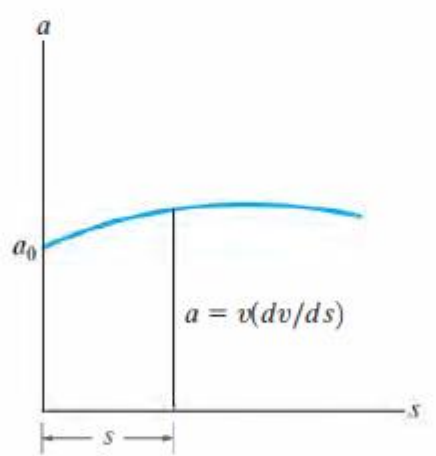
change in velocity = area under a–t graph



- **Rectilinear motion continued …**
- **Area under the a-s curve**



(a)

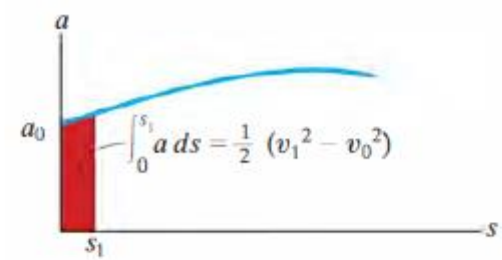


if you are given $v = f(s)$ or $a = g(s)$,

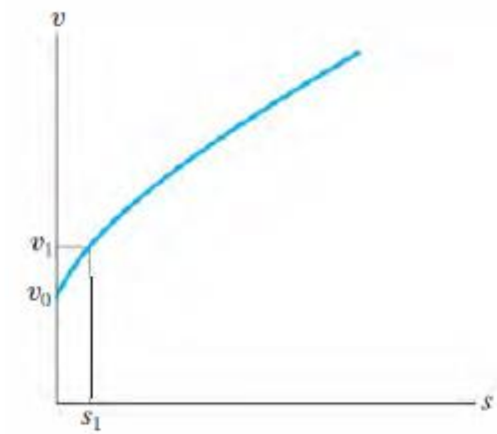
you can use $a\,ds = v\,dv$

$$\int_{v_1}^{v_2} v dv = \int_{s_1}^{s_2} a ds$$ **or**

$$\frac{1}{2}(v_2{}^2 - v_1{}^2) = (area\ under\ a - s\ curve$$



(a)



- **Rectilinear motion continued …**
- **Constant acceleration**

Velocity as a Function of Time. Integrate $a_c = dv/dt$, assuming that initially $v = v_0$ when $t = 0$.

$$\int_{v_0}^{v} dv = \int_0^t a_c \, dt$$

$$(\overset{+}{\rightarrow}) \qquad \boxed{\begin{array}{c} v = v_0 + a_c t \\ \text{Constant Acceleration} \end{array}} \qquad (12\text{–}4)$$

- **Rectilinear motion continued …**
- **Constant acceleration**

Position as a Function of Time. Integrate $v = ds/dt = v_0 + a_c t$, assuming that initially $s = s_0$ when $t = 0$.

$$\int_{s_0}^{s} ds = \int_0^t (v_0 + a_c t)\, dt$$

$$(\xrightarrow{+}) \qquad \boxed{s = s_0 + v_0 t + \tfrac{1}{2} a_c t^2} \qquad (12\text{–}5)$$

Constant Acceleration

- **Rectilinear motion continued …**
- **Constant acceleration**

Velocity as a Function of Position. Either solve for t in Eq. 12–4 and substitute into Eq. 12–5, or integrate $v\,dv = a_c\,ds$, assuming that initially $v = v_0$ at $s = s_0$.

$$\int_{v_0}^{v} v\,dv = \int_{s_0}^{s} a_c\,ds$$

$$(\overset{+}{\rightarrow}) \qquad \boxed{\begin{array}{c} v^2 = v_0^2 + 2a_c(s - s_0) \\ \text{Constant Acceleration} \end{array}} \qquad (12\text{–}6)$$

Caution: The foregoing equations have been integrated for constant acceleration only. A common mistake is to use these equations for problems involving variable acceleration, where they do not apply.

- **Rectilinear motion continued …**
- **Rockets**

During the time this rocket undergoes rectilinear motion, its altitude as a function of time can be measured and expressed as $s = s(t)$. Its velocity can then be found using $v = ds/dt$, and its acceleration can be determined from $a = dv/dt$.

- **Rectilinear motion continued …**
- **Acceleration given as a function of time, a=f(t)**
- **because** $a = \frac{dv}{dt} = \dot{v}$ **then**

$$\int_{v_0}^{v} dv = \int_0^t f(t)dt \qquad \text{or} \qquad v = v_0 + \int_0^t f(t)dt$$

- **Velocity as a function of time**

$$\int_{s_0}^{s} ds = \int_0^t vdt \qquad\qquad s = s_0 + \int_0^t vdt$$

- **Rectilinear motion continued …**
- **Acceleration given as a function of velocity, a=f(v)**
- **But** $a = \frac{dv}{dt} = \dot{v}$ **f(v) = dv/dt -> separating variables**
  - **dt=dv/f(v) therefore**

$$t = \int_0^t dt = \int_{v_0}^{v} \frac{dv}{f(v)}$$

- **Alternatively vdv=f(v)ds resulting into**

$$\int_{v_0}^{v} \frac{vdv}{f(v)} = \int_{s_0}^{s} ds \quad \textbf{or} \quad s = s_0 + \int_{v_0}^{v} \frac{vdv}{f(v)}$$

- **Rectilinear motion continued …**
- **Acceleration given as a function of displacement, a=f(s)**
- **Substituting into** vdv=ads **and integrating**

$$\int_{v_0}^{v} dv = \int_{s_0}^{s} f(s)ds \quad \textbf{or} \quad v^2 = {v_0}^2 + 2\int_{s_0}^{s} f(s)ds$$

Next we solve for v to give v=g(s), **a function of** s

Now we can substitute ds/dt **for** v **and separate variables and integrate**

$$\int_{s_0}^{s} \frac{ds}{g(s)} = \int_{0}^{t} dt \quad \textbf{or} \quad t = \int_{s_0}^{s} \frac{ds}{g(s)}$$

Sample Problem 2/1

The position coordinate of a particle which is confined to move along a straight line is given by $s = 2t^3 - 24t + 6$, where s is measured in meters from a convenient origin and t is in seconds. Determine (a) the time required for the particle to reach a velocity of 72 m/s from its initial condition at $t = 0$, (b) the acceleration of the particle when $v = 30$ m/s, and (c) the net displacement of the particle during the interval from $t = 1$ s to $t = 4$ s.

Solution. The velocity and acceleration are obtained by successive differentiation of s with respect to the time. Thus,

$[v = \dot{s}]$ $\qquad v = 6t^2 - 24 \text{ m/s}$

$[a = \dot{v}]$ $\qquad a = 12t \text{ m/s}^2$

(a) Substituting $v = 72$ m/s into the expression for v gives us $72 = 6t^2 - 24$, from which $t = \pm 4$ s. The negative root describes a mathematical solution for t before
① the initiation of motion, so this root is of no physical interest. Thus, the desired result is

$$t = 4 \text{ s} \qquad \textit{Ans.}$$

(b) Substituting $v = 30$ m/s into the expression for v gives $30 = 6t^2 - 24$, from which the positive root is $t = 3$ s, and the corresponding acceleration is

$$a = 12(3) = 36 \text{ m/s}^2 \qquad \textit{Ans.}$$

(c) The net displacement during the specified interval is

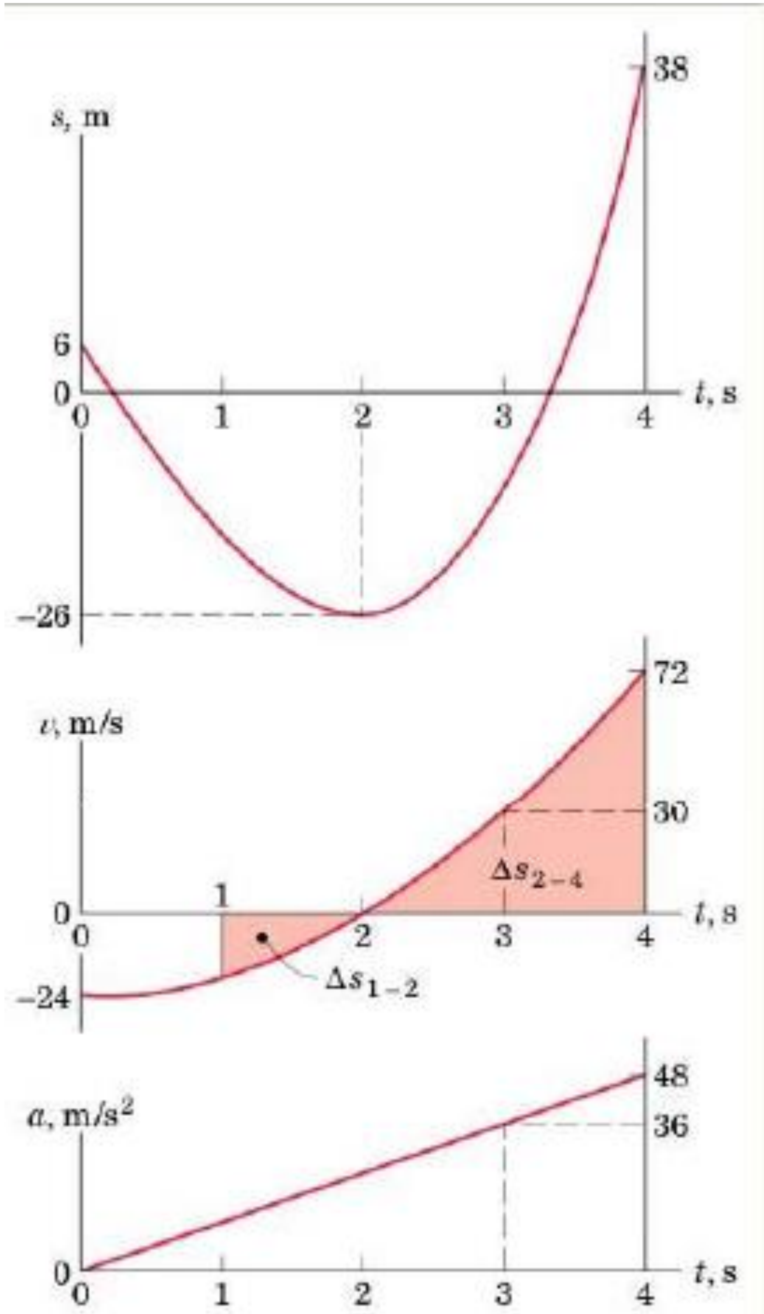
$$\Delta s = s_4 - s_1 \qquad \text{or}$$

$$\Delta s = [2(4^3) - 24(4) + 6] - [2(1^3) - 24(1) + 6]$$

$$= 54 \text{ m} \qquad \textit{Ans.}$$

② which represents the net advancement of the particle along the s-axis from the position it occupied at $t = 1$ s to its position at $t = 4$ s.

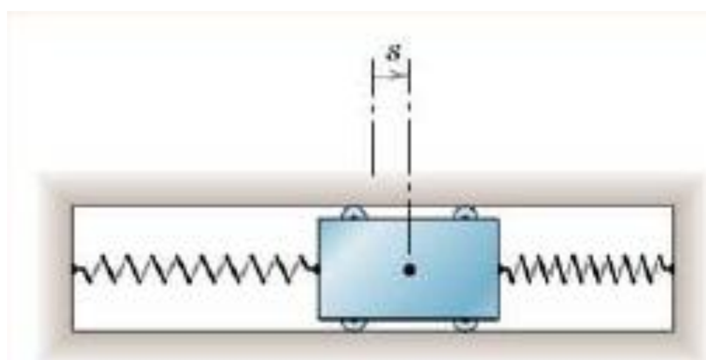
To help visualize the motion, the values of s, v, and a are plotted against the time t as shown. Because the area under the v-t curve represents displacement,
③ we see that the net displacement from $t = 1$ s to $t = 4$ s is the positive area Δs_{2-4} less the negative area Δs_{1-2}.



s, m
38
6
0
−26
0
1
2
3
4
t, s
v, m/s
72
30
1
0
0
2
3
4
t, s
−24
Δs_{2-4}
Δs_{1-2}
a, m/s²
48
36
0
0
1
2
3
4
t, s

Sample Problem 2/3

The spring-mounted slider moves in the horizontal guide with negligible friction and has a velocity v_0 in the s-direction as it crosses the mid-position where $s = 0$ and $t = 0$. The two springs together exert a retarding force to the motion of the slider, which gives it an acceleration proportional to the displacement but oppositely directed and equal to $a = -k^2 s$, where k is constant. (The constant is arbitrarily squared for later convenience in the form of the expressions.) Determine the expressions for the displacement s and velocity v as functions of the time t.



Solution I. Since the acceleration is specified in terms of the displacement, the differential relation $v\,dv = a\,ds$ may be integrated. Thus,

① $$\int v\,dv = \int -k^2 s\,ds + C_1 \text{ a constant,} \quad \text{or} \quad \frac{v^2}{2} = -\frac{k^2 s^2}{2} + C_1$$

When $s = 0$, $v = v_0$, so that $C_1 = v_0{}^2/2$, and the velocity becomes

$$v = +\sqrt{v_0{}^2 - k^2 s^2}$$

The plus sign of the radical is taken when v is positive (in the plus s-direction). This last expression may be integrated by substituting $v = ds/dt$. Thus,

Helpful Hint

① We have used an indefinite integral here and evaluated the constant of integration. For practice, obtain the same results by using the definite integral with the appropriate limits.

② $$\int \frac{ds}{\sqrt{v_0^2 - k^2 s^2}} = \int dt + C_2 \text{ a constant,} \quad \text{or} \quad \frac{1}{k}\sin^{-1}\frac{ks}{v_0} = t + C_2$$

With the requirement of $t = 0$ when $s = 0$, the constant of integration becomes $C_2 = 0$, and we may solve the equation for s so that

$$s = \frac{v_0}{k}\sin kt \qquad \textit{Ans.}$$

The velocity is $v = \dot{s}$, which gives

$$v = v_0 \cos kt \qquad \textit{Ans.}$$

Helpful Hint

② Again try the definite integral here as above.

Solution II. Since $a = \ddot{s}$, the given relation may be written at once as

$$\ddot{s} + k^2 s = 0$$

This is an ordinary linear differential equation of second order for which the solution is well known and is

$$s = A \sin Kt + B \cos Kt$$

where A, B, and K are constants. Substitution of this expression into the differential equation shows that it satisfies the equation, provided that $K = k$. The velocity is $v = \dot{s}$, which becomes

$$v = Ak \cos kt - Bk \sin kt$$

The initial condition $v = v_0$ when $t = 0$ requires that $A = v_0/k$, and the condition $s = 0$ when $t = 0$ gives $B = 0$. Thus, the solution is

③ $$s = \frac{v_0}{k} \sin kt \qquad \text{and} \qquad v = v_0 \cos kt \qquad \textit{Ans.}$$

Helpful Hints

③ This motion is called *simple harmonic motion* and is characteristic of all oscillations where the restoring force, and hence the acceleration, is proportional to the displacement but opposite in sign.

2/3 Plane Curvilinear Motion

We now treat the motion of a particle along a curved path which lies in a single plane. This motion is a special case of the more general three-dimensional motion introduced in Art. 2/1 and illustrated in Fig. 2/1. If we let the plane of motion be the *x*-*y* plane, for instance, then the coordinates z and ϕ of Fig. 2/1 are both zero, and R becomes the same as r. As mentioned previously, the vast majority of the motions of points or particles encountered in engineering practice can be represented as plane motion.

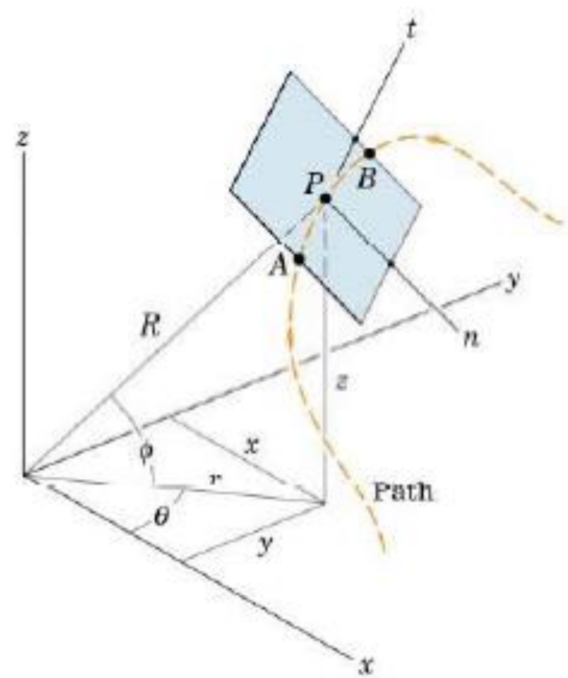


Figure 2/1

Important!

Before pursuing the description of plane curvilinear motion in any specific set of coordinates, we will first use vector analysis to describe the motion, since the results will be independent of any particular coordinate system. What follows in this article constitutes one of the most basic concepts in dynamics, namely, the *time derivative of a vector*. Much analysis in dynamics utilizes the time rates of change of vector quantities. You are therefore well advised to master this topic at the outset because you will have frequent occasion to use it.

Consider now the continuous motion of a particle along a plane curve as represented in Fig. 2/5. At time t the particle is at position A, which is located by the *position vector* $\mathbf{r}$ measured from some convenient fixed origin O. If both the magnitude and direction of $\mathbf{r}$ are known at time t, then the position of the particle is completely specified. At time $t + \Delta t$, the particle is at A', located by the position vector $\mathbf{r} + \Delta\mathbf{r}$. We note, of course, that this combination is vector addition and not scalar addition. The *displacement* of the particle during time Δt is the vector $\Delta\mathbf{r}$ which represents the vector change of position and is clearly independent of the choice of origin. If an origin were chosen at some different location, the position vector $\mathbf{r}$ would be changed, but $\Delta\mathbf{r}$ would be unchanged. The *distance* actually traveled by the particle as it moves along the path from A to A' is the scalar length Δs measured along the path. Thus, we distinguish between the vector displacement $\Delta\mathbf{r}$ and the scalar distance Δs.

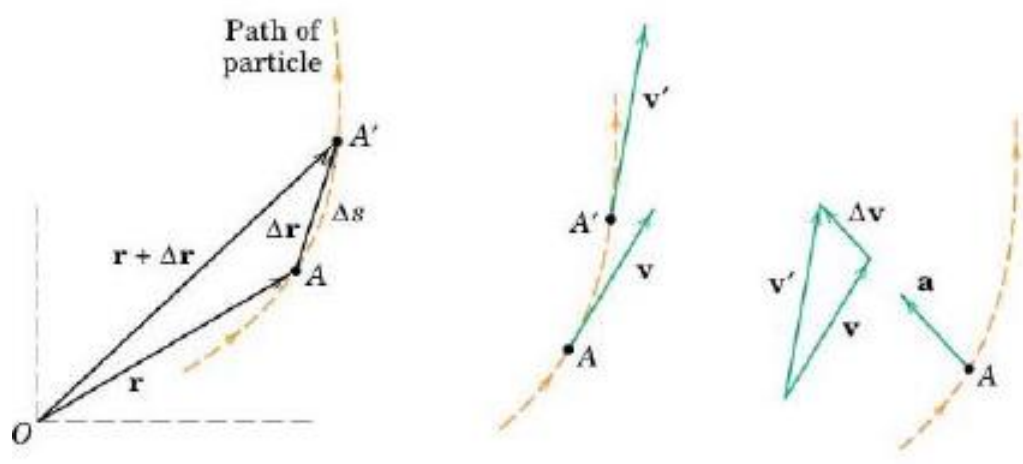


Figure 2/5

Velocity

The *average velocity* of the particle between A and A' is defined as $\mathbf{v}_{av} = \Delta\mathbf{r}/\Delta t$, which is a vector whose direction is that of $\Delta\mathbf{r}$ and whose magnitude is the magnitude of $\Delta\mathbf{r}$ divided by Δt. The average speed of

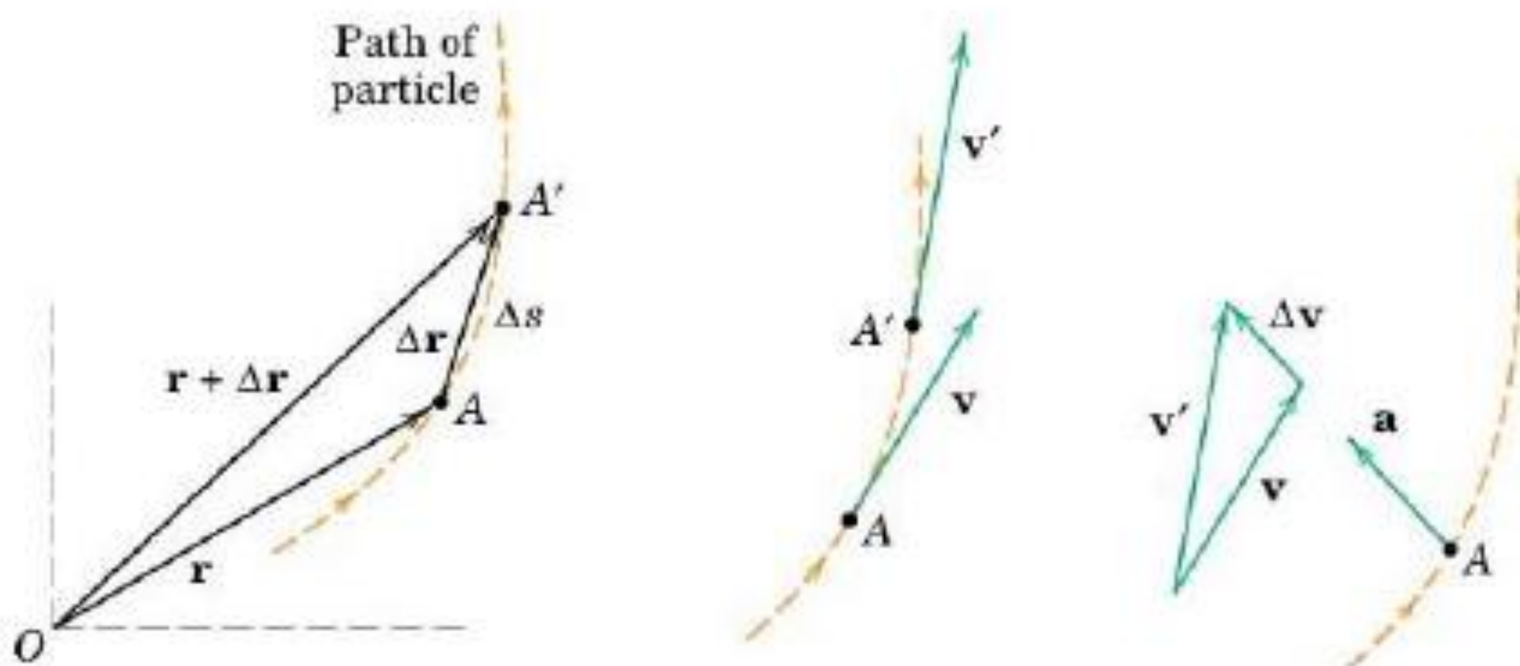


Figure 2/5

the particle between A and A' is the scalar quotient $\Delta s/\Delta t$. Clearly, the magnitude of the average velocity and the speed approach one another as the interval Δt decreases and A and A' become closer together.

The *instantaneous velocity* $\mathbf{v}$ of the particle is defined as the limiting value of the average velocity as the time interval approaches zero. Thus,

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t}$$

We observe that the direction of $\Delta \mathbf{r}$ approaches that of the tangent to the path as Δt approaches zero and, thus, the velocity $\mathbf{v}$ is always a vector tangent to the path.

We now extend the basic definition of the derivative of a scalar quantity to include a vector quantity and write

$$\boxed{\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}} \qquad \textbf{(2/4)}$$

The derivative of a vector is itself a vector having both a magnitude and a direction. The magnitude of $\mathbf{v}$ is called the *speed* and is the scalar

$$v = |\mathbf{v}| = \frac{ds}{dt} = \dot{s}$$

At this point we make a careful distinction between the *magnitude of the derivative* and the *derivative of the magnitude*. The magnitude of the derivative can be written in any one of the several ways $|d\mathbf{r}/dt| = |\dot{\mathbf{r}}| = \dot{s} = |\mathbf{v}| = v$ and represents the magnitude of the velocity, or the speed, of the particle. On the other hand, the derivative of the magnitude is written $d|\mathbf{r}|/dt = dr/dt = \dot{r}$, and represents the rate at which the length of the position vector $\mathbf{r}$ is changing. Thus, these two derivatives have two entirely different meanings, and we must be extremely careful to distinguish between them in our thinking and in our notation. For this and other reasons, you are urged to adopt a consistent notation for handwritten work for all vector quantities to distinguish them from scalar quantities. For simplicity the underline $\underline{v}$ is recommended. Other handwritten symbols such as $\vec{v}$, $\underset{\sim}{v}$, and $\hat{v}$ are sometimes used.

With the concept of velocity as a vector established, we return to Fig. 2/5 and denote the velocity of the particle at A by the tangent vector $\mathbf{v}$ and the velocity at A' by the tangent $\mathbf{v}'$. Clearly, there is a vector change in the velocity during the time Δt. The velocity $\mathbf{v}$ at A plus (vectorially) the change $\Delta\mathbf{v}$ must equal the velocity at A', so we can write $\mathbf{v}' - \mathbf{v} = \Delta\mathbf{v}$. Inspection of the vector diagram shows that $\Delta\mathbf{v}$ depends both on the change in magnitude (length) of $\mathbf{v}$ and on the change in direction of $\mathbf{v}$. These two changes are fundamental characteristics of the derivative of a vector.

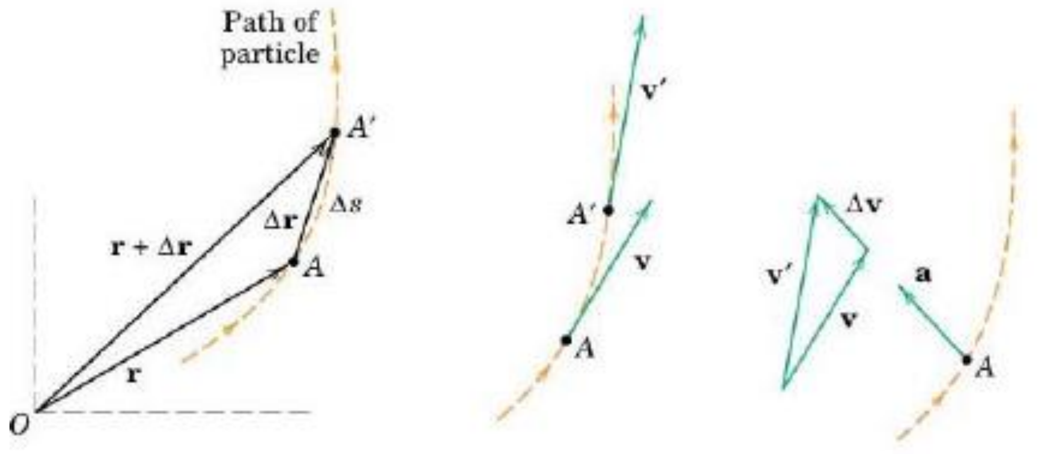


Figure 2/5

Acceleration

The *average acceleration* of the particle between A and A' is defined as $\Delta\mathbf{v}/\Delta t$, which is a vector whose direction is that of $\Delta\mathbf{v}$. The magnitude of this average acceleration is the magnitude of $\Delta\mathbf{v}$ divided by Δt.

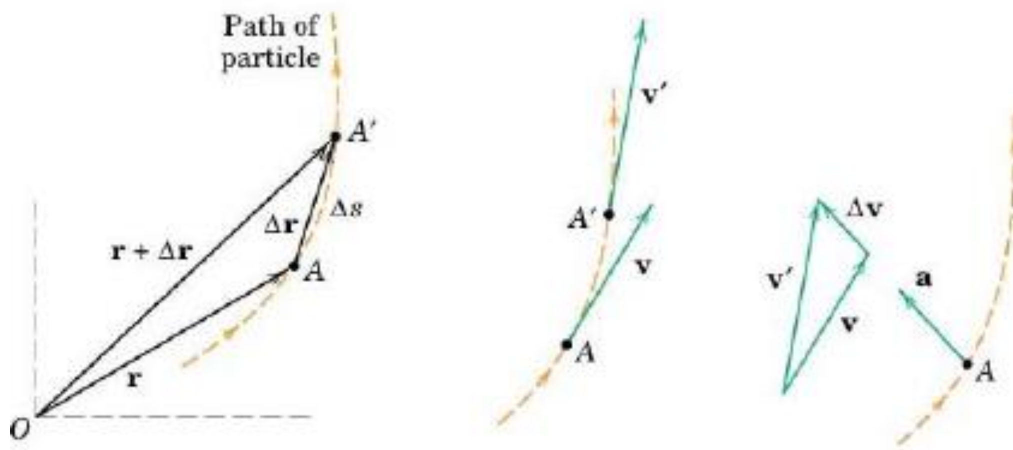


Figure 2/5

The *instantaneous acceleration* $\mathbf{a}$ of the particle is defined as the limiting value of the average acceleration as the time interval approaches zero. Thus,

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{v}}{\Delta t}$$

By definition of the derivative, then, we write

$$\boxed{\mathbf{a} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}}} \qquad \textbf{(2/5)}$$

As the interval Δt becomes smaller and approaches zero, the direction of the change $\Delta \mathbf{v}$ approaches that of the differential change $d\mathbf{v}$ and, thus, of $\mathbf{a}$. The acceleration $\mathbf{a}$, then, includes the effects of both the change in magnitude of $\mathbf{v}$ and the change of direction of $\mathbf{v}$. It is apparent, in general, that the direction of the acceleration of a particle in curvilinear motion is neither tangent to the path nor normal to the path. We do observe, however, that the acceleration component which is normal to the path points toward the center of curvature of the path.

Visualization of Motion

A further approach to the visualization of acceleration is shown in Fig. 2/6, where the position vectors to three arbitrary positions on the path of the particle are shown for illustrative purpose. There is a velocity vector tangent to the path corresponding to each position vector, and the relation is $\mathbf{v} = \dot{\mathbf{r}}$. If these velocity vectors are now plotted from some arbitrary point C, a curve, called the *hodograph*, is formed. The derivatives of these velocity vectors will be the acceleration vectors $\mathbf{a} = \dot{\mathbf{v}}$ which are tangent to the hodograph. We see that the acceleration has the same relation to the velocity as the velocity has to the position vector.

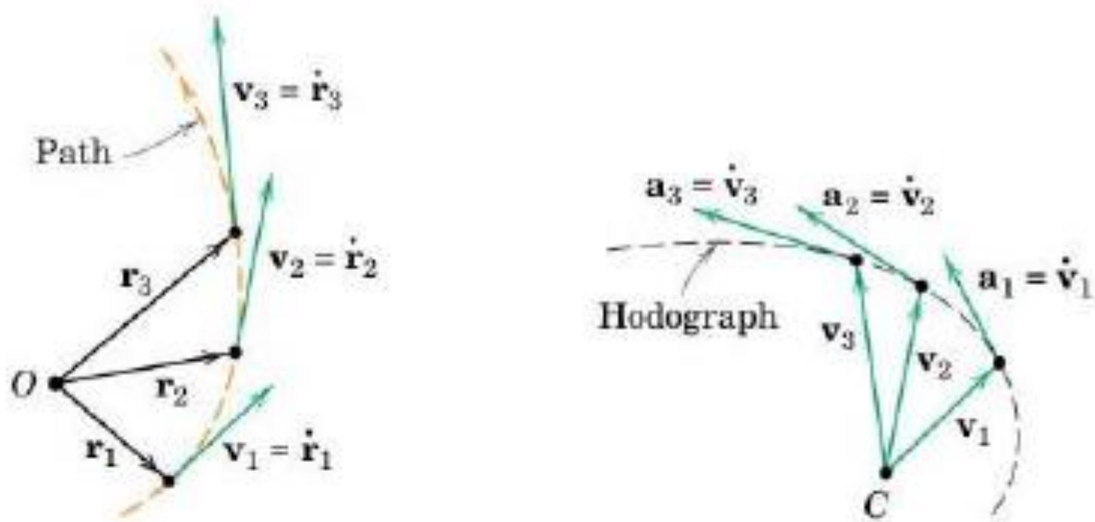


Figure 2/6

• Rules for vector differentiation

The geometric portrayal of the derivatives of the position vector **r** and velocity vector **v** in Fig. 2/5 can be used to describe the derivative of any vector quantity with respect to t or with respect to any other scalar variable. Now that we have used the definitions of velocity and acceleration to introduce the concept of the derivative of a vector, it is important to establish the rules for differentiating vector quantities. These rules

are the same as for the differentiation of scalar quantities, except for the case of the cross product where the order of the terms must be preserved. These rules are covered in Art. C/7 of Appendix C and should be reviewed at this point.

9. ***Derivatives of vectors*** obey the same rules as they do for scalars.

$$\frac{d\mathbf{P}}{dt} = \dot{\mathbf{P}} = \dot{P}_x\mathbf{i} + \dot{P}_y\mathbf{j} + \dot{P}_z\mathbf{k}$$

$$\frac{d(\mathbf{P}u)}{dt} = \mathbf{P}\dot{u} + \dot{\mathbf{P}}u$$

$$\frac{d(\mathbf{P}\cdot\mathbf{Q})}{dt} = \mathbf{P}\cdot\dot{\mathbf{Q}} + \dot{\mathbf{P}}\cdot\mathbf{Q}$$

$$\frac{d(\mathbf{P}\times\mathbf{Q})}{dt} = \mathbf{P}\times\dot{\mathbf{Q}} + \dot{\mathbf{P}}\times\mathbf{Q}$$

- Plane motion choice of coordinates

Three different coordinate systems are commonly used for describing the vector relationships for curvilinear motion of a particle in a plane: rectangular coordinates, normal and tangential coordinates, and polar coordinates. An important lesson to be learned from the study of these coordinate systems is the proper choice of a reference system for a given problem. This choice is usually revealed by the manner in which the motion is generated or by the form in which the data are specified. Each of the three coordinate systems will now be developed and illustrated.

2/4 Rectangular Coordinates (x-y)

This system of coordinates is particularly useful for describing motions where the x- and y-components of acceleration are independently generated or determined. The resulting curvilinear motion is then obtained by a vector combination of the x- and y-components of the position vector, the velocity, and the acceleration.

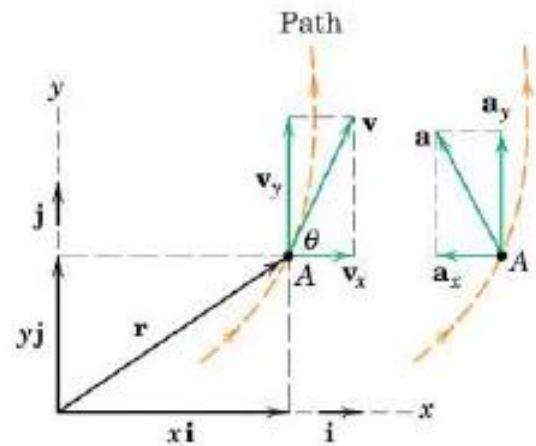


Figure 2/7

Vector Representation

The particle path of Fig. 2/5 is shown again in Fig. 2/7 along with x- and y-axes. The position vector $\mathbf{r}$, the velocity $\mathbf{v}$, and the acceleration $\mathbf{a}$ of the particle as developed in Art. 2/3 are represented in Fig. 2/7 together with their x- and y-components. With the aid of the unit vectors $\mathbf{i}$ and $\mathbf{j}$, we can write the vectors $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$ in terms of their x- and y-components. Thus,

$$\boxed{\begin{aligned} \mathbf{r} &= x\mathbf{i} + y\mathbf{j} \\ \mathbf{v} = \dot{\mathbf{r}} &= \dot{x}\mathbf{i} + \dot{y}\mathbf{j} \\ \mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} &= \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} \end{aligned}} \qquad \textbf{(2/6)}$$

Note

As we differentiate with respect to time, we observe that the time derivatives of the unit vectors are zero because their magnitudes and directions remain constant. The scalar values of the components of **v** and **a** are merely $v_x = \dot{x}$, $v_y = \dot{y}$ and $a_x = \dot{v}_x = \ddot{x}$, $a_y = \dot{v}_y = \ddot{y}$. (As drawn in Fig. 2/7, a_x is in the negative x-direction, so that $\ddot{x}$ would be a negative number.)

As observed previously, the direction of the velocity is always tangent to the path, and from the figure it is clear that

$$v^2 = {v_x}^2 + {v_y}^2 \qquad v = \sqrt{{v_x}^2 + {v_y}^2} \qquad \tan\theta = \frac{v_y}{v_x}$$

$$a^2 = {a_x}^2 + {a_y}^2 \qquad a = \sqrt{{a_x}^2 + {a_y}^2}$$

If the angle θ is measured counterclockwise from the x-axis to $\mathbf{v}$ for the configuration of axes shown, then we can also observe that $dy/dx = \tan\theta = v_y/v_x$.

If the coordinates x and y are known independently as functions of time, $x = f_1(t)$ and $y = f_2(t)$, then for any value of the time we can combine them to obtain **r**. Similarly, we combine their first derivatives $\dot{x}$ and $\dot{y}$ to obtain **v** and their second derivatives $\ddot{x}$ and $\ddot{y}$ to obtain **a**. On the other hand, if the acceleration components a_x and a_y are given as functions of the time, we can integrate each one separately with respect to time, once to obtain v_x and v_y and again to obtain $x = f_1(t)$ and $y = f_2(t)$. Elimination of the time t between these last two parametric equations gives the equation of the curved path $y = f(x)$.

From the foregoing discussion we can see that the rectangular-coordinate representation of curvilinear motion is merely the superposition of the components of two simultaneous rectilinear motions in the x- and y-directions. Therefore, everything covered in Art. 2/2 on rectilinear motion can be applied separately to the x-motion and to the y-motion.

- Examples

Projectile Motion

An important application of two-dimensional kinematic theory is the problem of projectile motion. For a first treatment of the subject, we neglect aerodynamic drag and the curvature and rotation of the earth, and we assume that the altitude change is small enough so that the acceleration due to gravity can be considered constant. With these assumptions, rectangular coordinates are useful for the trajectory analysis.

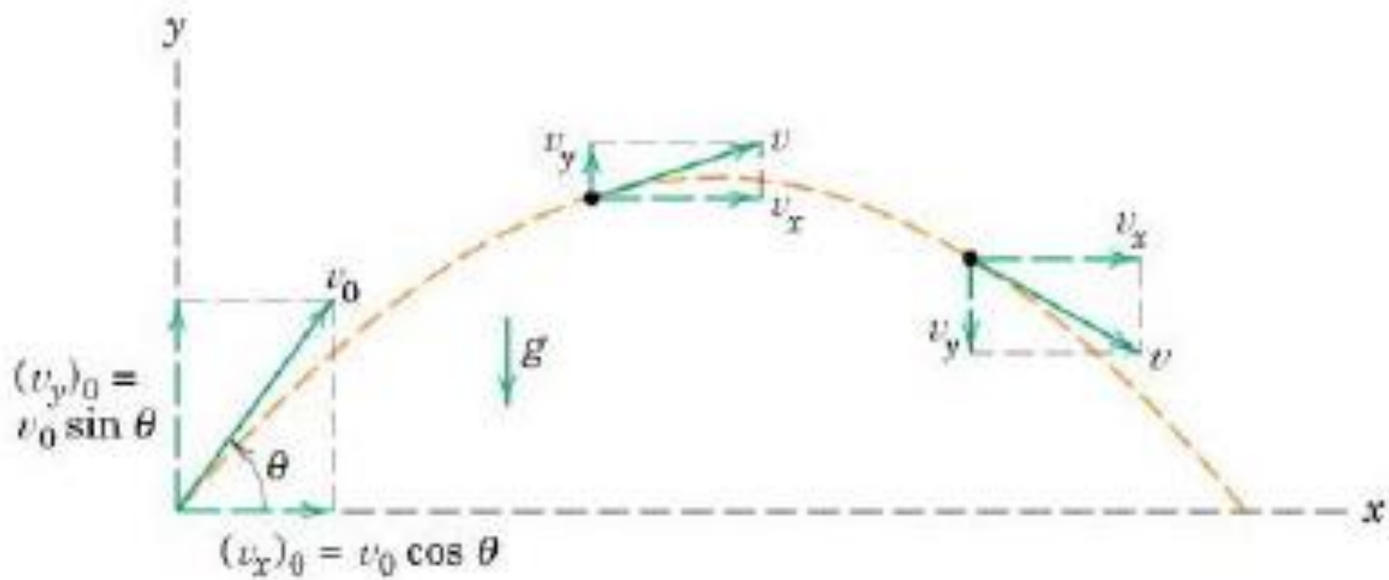


Figure 2/8

For the axes shown in Fig. 2/8, the acceleration components are

$$a_x = 0 \qquad a_y = -g$$

Integration of these accelerations follows the results obtained previously in Art. 2/2a for constant acceleration and yields

$$v_x = (v_x)_0 \qquad v_y = (v_y)_0 - gt$$

$$x = x_0 + (v_x)_0 t \qquad y = y_0 + (v_y)_0 t - \frac{1}{2} g t^2$$

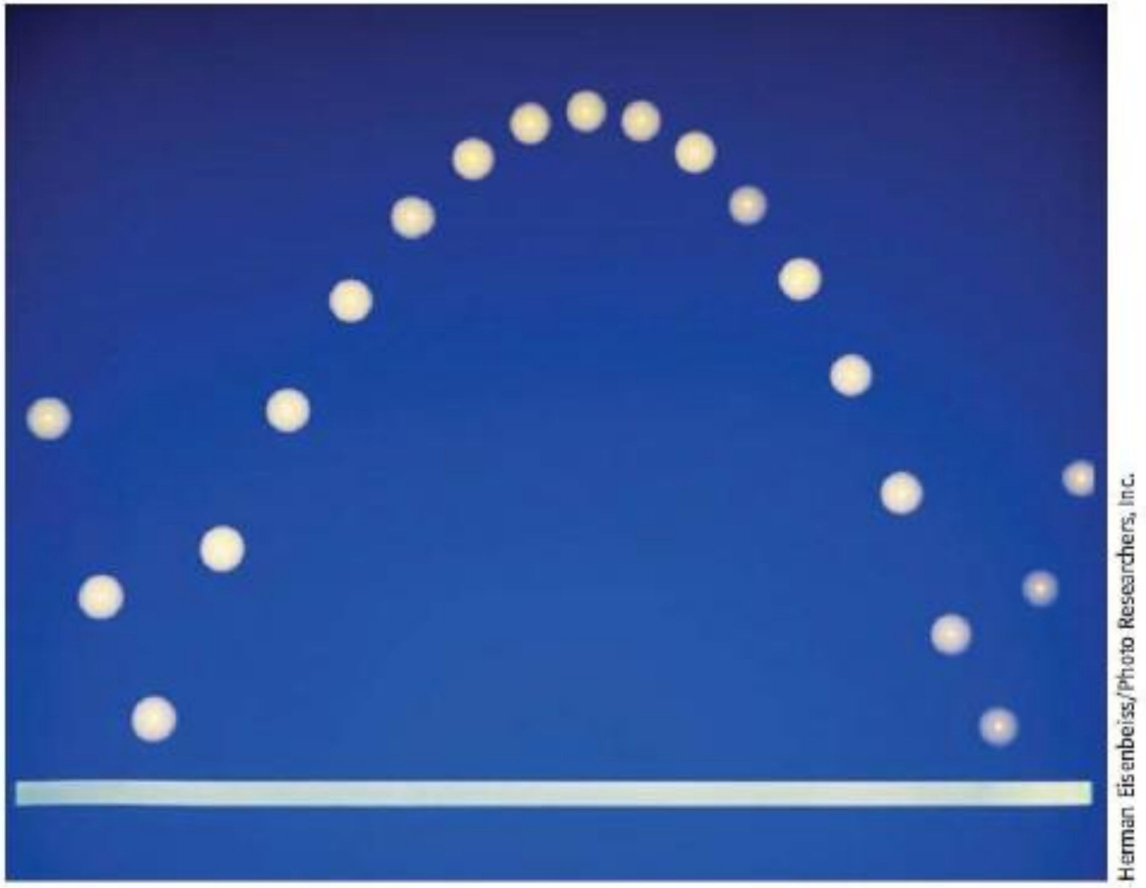
$$v_y{}^2 = (v_y)_0{}^2 - 2g(y - y_0)$$

In all these expressions, the subscript zero denotes initial conditions, frequently taken as those at launch where, for the case illustrated, $x_0 = y_0 = 0$. Note that the quantity g is taken to be positive throughout this text.

We can see that the x- and y-motions are independent for the simple projectile conditions under consideration. Elimination of the time t between the x- and y-displacement equations shows the path to be parabolic (see Sample Problem 2/6). If we were to introduce a drag force which depends on the speed squared (for example), then the x- and y-motions would be coupled (interdependent), and the trajectory would be nonparabolic.

- However

When the projectile motion involves large velocities and high altitudes, to obtain accurate results we must account for the shape of the projectile, the variation of g with altitude, the variation of the air density with altitude, and the rotation of the earth. These factors introduce considerable complexity into the motion equations, and numerical integration of the acceleration equations is usually necessary.



This stroboscopic photograph of a bouncing ping-pong ball suggests not only the parabolic nature of the path, but also the fact that the speed is lower near the apex.

Sample Problem 2/5

The curvilinear motion of a particle is defined by $v_x = 50 - 16t$ and $y = 100 - 4t^2$, where v_x is in meters per second, y is in meters, and t is in seconds. It is also known that $x = 0$ when $t = 0$. Plot the path of the particle and determine its velocity and acceleration when the position $y = 0$ is reached.

Solution. The x-coordinate is obtained by integrating the expression for v_x, and the x-component of the acceleration is obtained by differentiating v_x. Thus,

$$\left[\int dx = \int v_x\,dt\right] \qquad \int_0^x dx = \int_0^t (50 - 16t)\,dt \qquad x = 50t - 8t^2 \text{ m}$$

$$[a_x = \dot{v}_x] \qquad a_x = \frac{d}{dt}(50 - 16t) \qquad a_x = -16 \text{ m/s}^2$$

The y-components of velocity and acceleration are

$$[v_y = \dot{y}] \qquad v_y = \frac{d}{dt}(100 - 4t^2) \qquad v_y = -8t \text{ m/s}$$

$$[a_y = \dot{v}_y] \qquad a_y = \frac{d}{dt}(-8t) \qquad a_y = -8 \text{ m/s}^2$$

We now calculate corresponding values of x and y for various values of t and plot x against y to obtain the path as shown.

When $y = 0$, $0 = 100 - 4t^2$, so $t = 5$ s. For this value of the time, we have

$$v_x = 50 - 16(5) = -30 \text{ m/s}$$

$$v_y = -8(5) = -40 \text{ m/s}$$

$$v = \sqrt{(-30)^2 + (-40)^2} = 50 \text{ m/s}$$

$$a = \sqrt{(-16)^2 + (-8)^2} = 17.89 \text{ m/s}^2$$

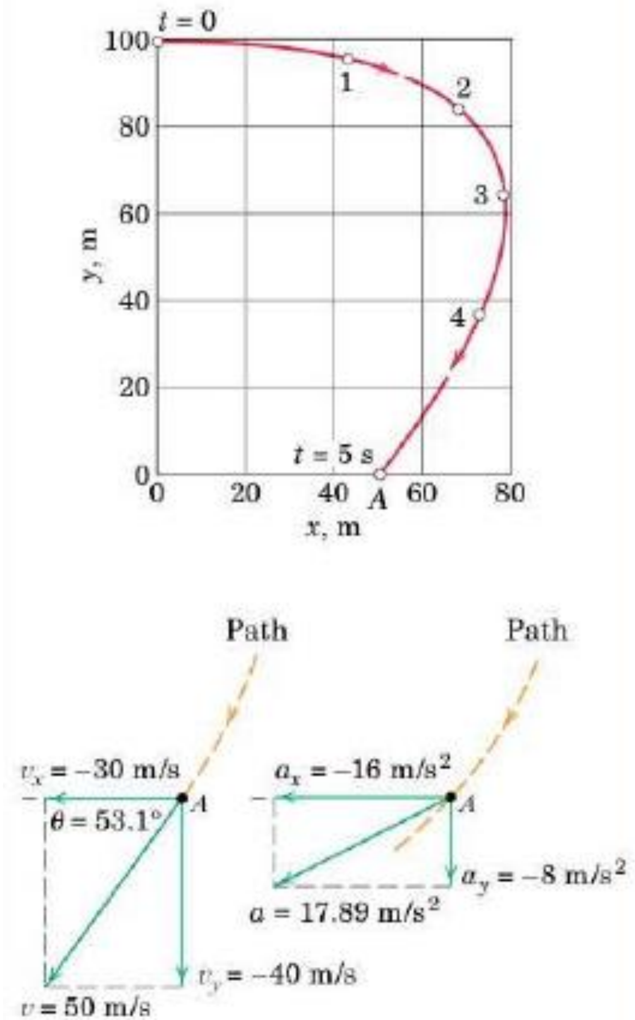
The velocity and acceleration components and their resultants are shown on the separate diagrams for point A, where $y = 0$. Thus, for this condition we may write

$$\mathbf{v} = -30\mathbf{i} - 40\mathbf{j} \text{ m/s} \qquad \textit{Ans.}$$

$$\mathbf{a} = -16\mathbf{i} - 8\mathbf{j} \text{ m/s}^2 \qquad \textit{Ans.}$$

Helpful Hint

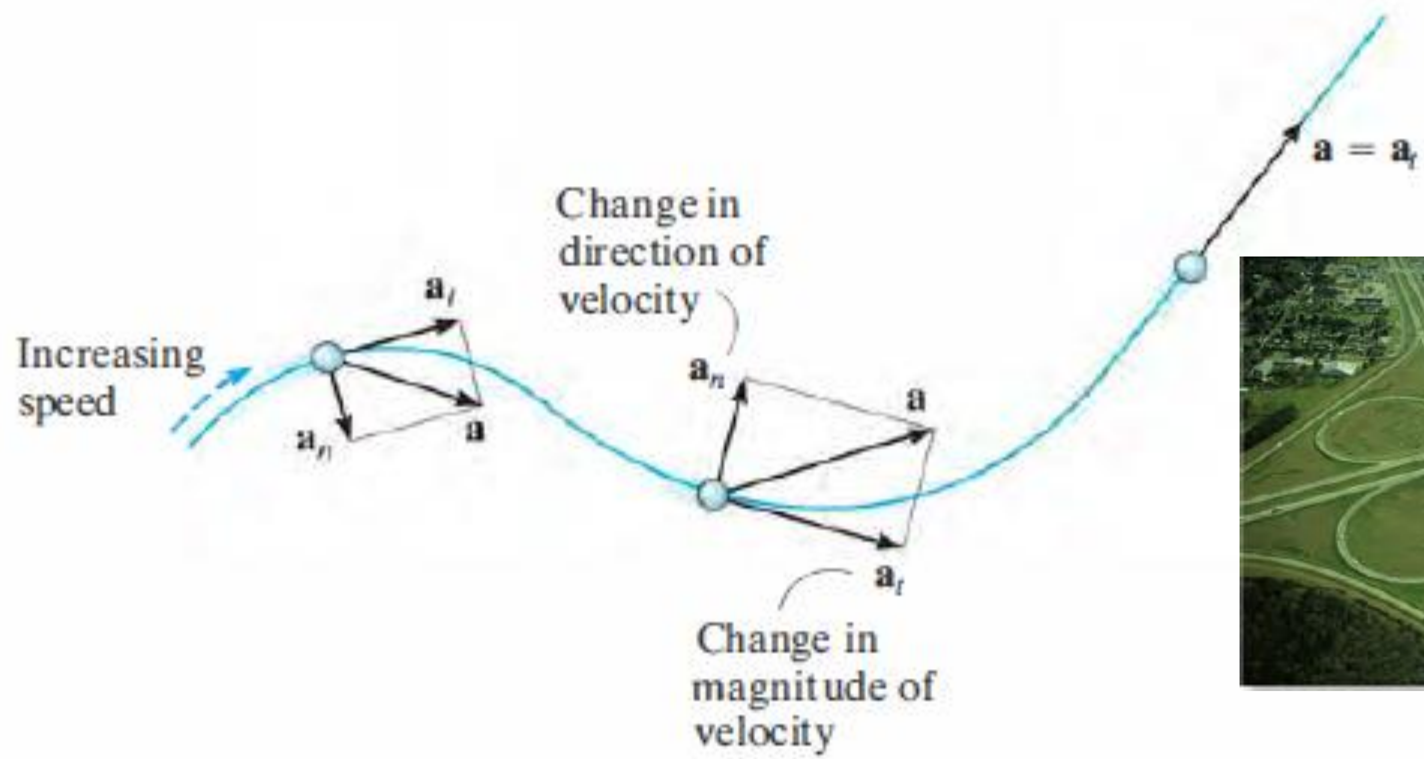
We observe that the velocity vector lies along the tangent to the path as it should, but that the acceleration vector is not tangent to the path. Note especially that the acceleration vector has a component that points toward the inside of the curved path. We concluded from our diagram in Fig. 2/5 that it is impossible for the acceleration to have a component that points toward the outside of the curve.

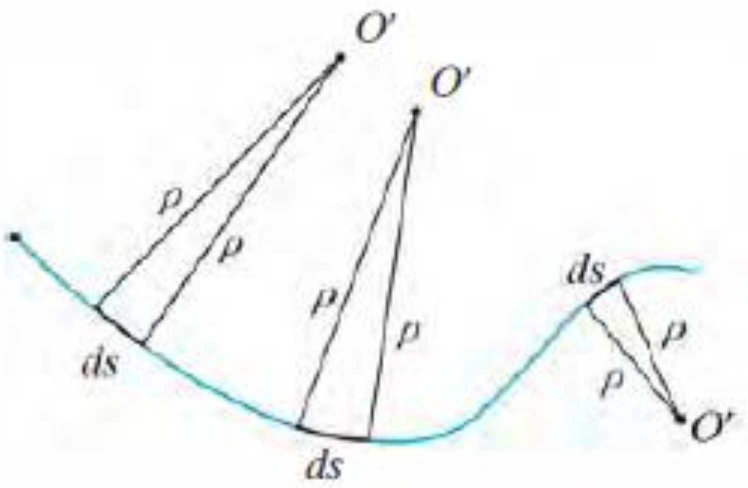


2.5 Normal and Tangential Coordinates.

- Sometimes the motion of the particle is constrained on a path that is best described using normal and tangential coordinates.

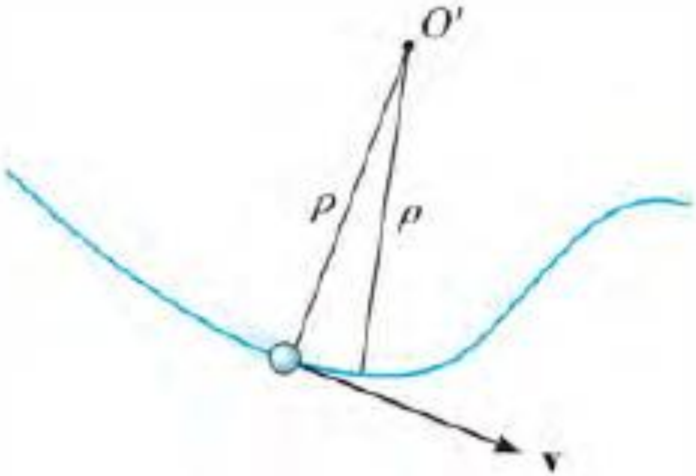
- For these coordinates the reference coordinate (origin) is moving with the particle





Radius of curvature

(b)



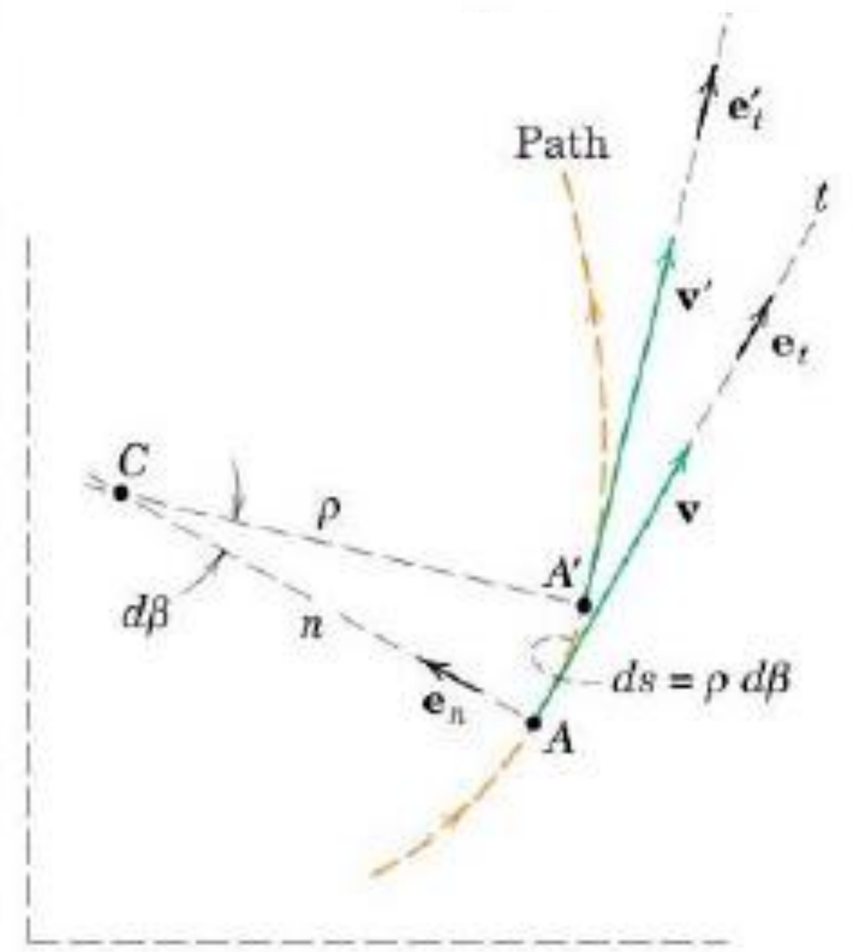
Velocity

(c)

- These axes can be used such as describing the motion of an aircraft where the origin is at the center of mass.

- ✓ **The tangential direction along longitudinal axis of the aircraft**
- ✓ **The normal axis along the length of the wings**

Velocity and Acceleration



$$v = \frac{ds}{dt} = \rho \frac{d\beta}{dt}$$

because $ds = \rho d\beta$

therefore

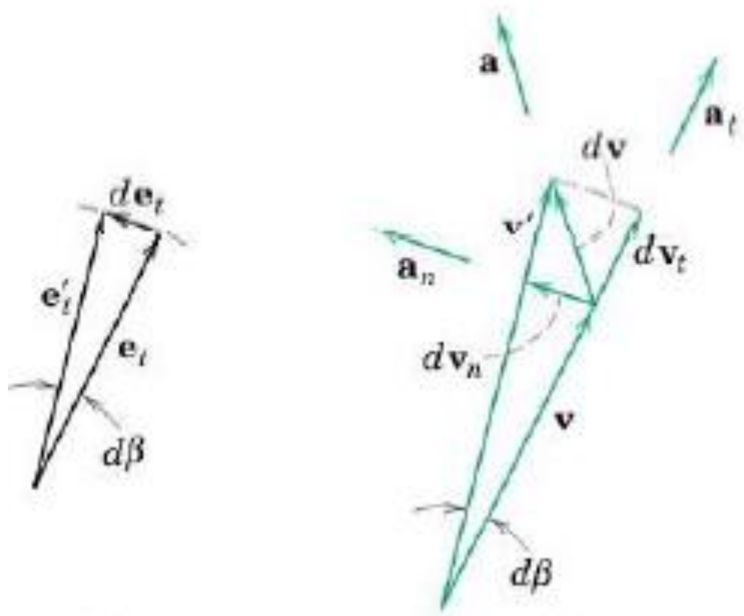
$$\boxed{\boldsymbol{v} = v\boldsymbol{e}_t = \rho\dot{\beta}\,\boldsymbol{e}_t}$$

and

$$\boxed{\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{d(v\boldsymbol{e}_t)}{dt} = v\dot{\boldsymbol{e}}_t + \dot{v}\boldsymbol{e}_t}$$

Where the unit vector e_t now has a derivative because its direction changes.

.



$$d\boldsymbol{e}_t = \boldsymbol{e}_t d\beta$$

$$|d\boldsymbol{e}_t| = |\boldsymbol{e}_t| d\beta = d\beta$$

$$\dot{\boldsymbol{e}}_t = \dot{\beta}\boldsymbol{e}_n$$

Therefore $\boldsymbol{a} = \dfrac{d\boldsymbol{v}}{dt} = \dfrac{d(v\boldsymbol{e}_t)}{dt} = v\dot{\boldsymbol{e}}_t + \dot{v}\boldsymbol{e}_t$ **with** $\boldsymbol{v} = v\boldsymbol{e}_t = \rho\dot{\beta}\boldsymbol{e}_t$

which gives $\boxed{\boldsymbol{a} = \dfrac{v^2}{\rho}\boldsymbol{e}_n + \dot{v}\boldsymbol{e}_t}$

.

Therefore

$$\boxed{\boldsymbol{a} = \frac{v^2}{\rho}\boldsymbol{e}_n + \dot{v}\boldsymbol{e}_t}$$

where

$$a_n = \frac{v^2}{\rho} = \rho\dot{\beta}^2 = v\dot{\beta}$$

$$a_t = \dot{v} = \ddot{s}$$

$$a = \sqrt{{a_n}^2 + {a_t}^2}$$

2.5

Normal Acceleration.

- The normal component of acceleration is the result of the time rate of change in the *direction* of the velocity. This component is *always* directed toward the center of curvature of the path, i.e., along the positive n axis.
- The magnitude of this component is determined from

$$a_n = \frac{v^2}{\rho}$$

- If the path is expressed as $y = f(x)$, the radius of curvature ρ at any point on the path is determined from the equation

$$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{|d^2y/dx^2|}$$

The derivation of this result is given in any standard calculus text.

2.5

Graphical visualization of the acceleration and the trajectory path

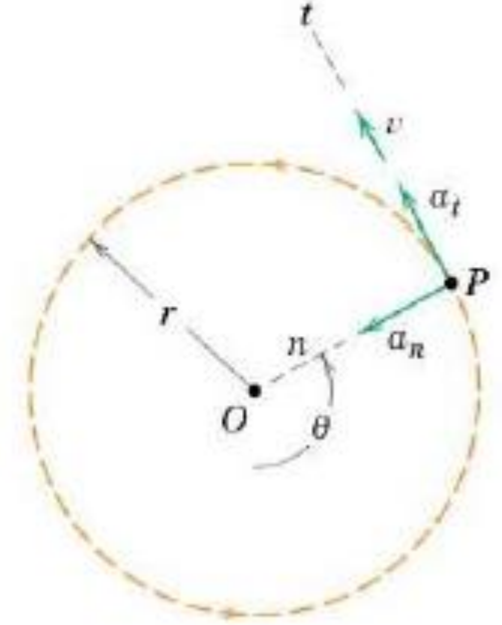
2.5

Special Case: Circular Motion

$$v = r\dot{\theta}$$

$$a_n = \frac{v^2}{r} = r\dot{\theta}^2 = v\dot{\theta}$$

$$a_t = \dot{v} = r\ddot{\theta}$$



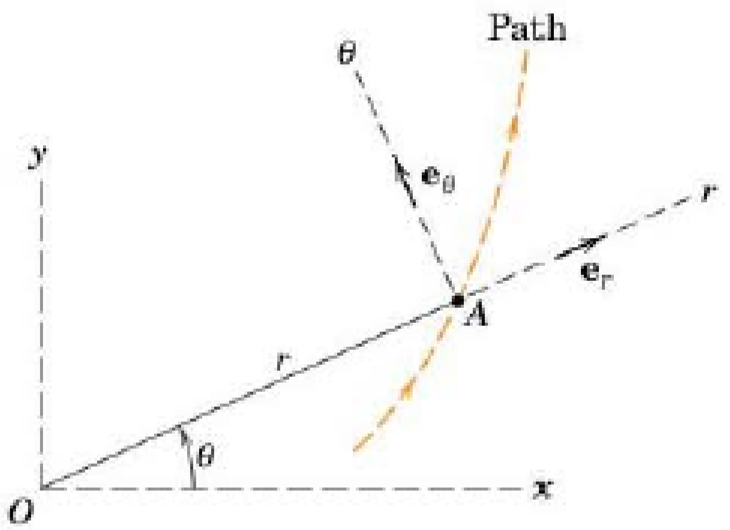
2.5

Sample Problems 2/7

Sample Problems 2/8: Go through

2.6 Polar Coordinates.

- We now consider Plane Curvilinear motion using Polar Coordinates.
- Useful when a motion is constrained through the control of a radial distance and an angular position.



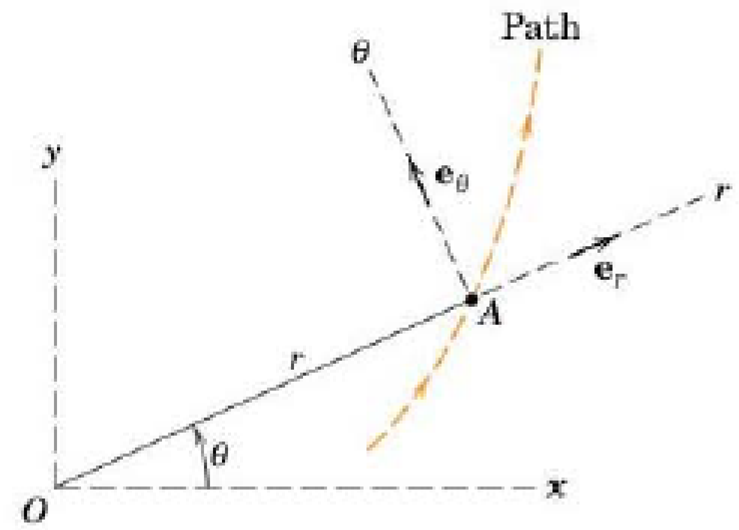
Polar Coordinates continued ...

- We can specify the location of the particle shown in the figure using:

- a radial coordinate r, which extends outward from the fixed origin 0 to the particle, and

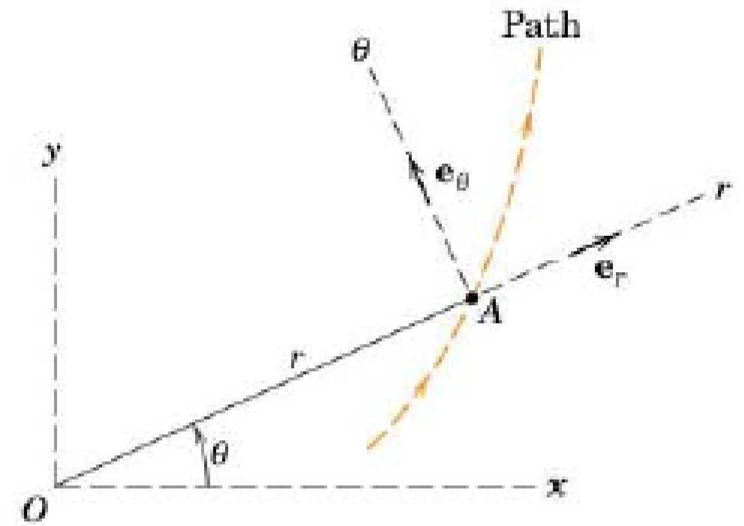
- a transverse coordinate θ, which is the counterclockwise angle between a fixed reference line and the r axis.

- The angle is generally measured in degrees or radians



2.6 Polar Coordinates.

- Position vector of particle $\boldsymbol{r} = r\boldsymbol{e}_r$
- We differentiate this vector with respect to time to get $\boldsymbol{v} = \dot{\boldsymbol{r}}$ and $\boldsymbol{a} = \dot{\boldsymbol{v}}$
- To successfully determine the derivatives we need time derivatives of both unit vectors
- This is exactly what we did to get the derivative of the tangential unit vector et



Polar coordinates continued ..

- The positive directions of the r and θ coordinates are defined by the unit vectors e_r and e_θ, respectively.

- Here e_r is in the direction of increasing r when θ is held fixed, and e_θ is in a direction of increasing e when r is held fixed.

- Note that these directions are perpendicular to one another.

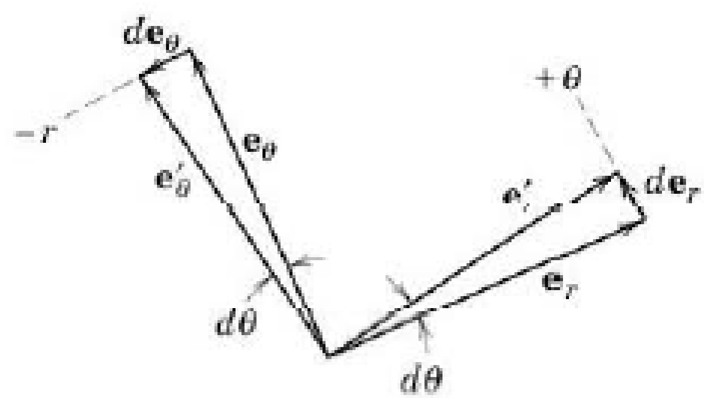
Polar coordinates continued ..

- During time dt the coordinate directions rotate through angle $d\theta$
- And the unit vectors rotate through same angle
- Note that the vector change de_r is in the plus θ – direction and that de_θ is in the minus r – direction
- Their magnitudes in the limit is unit vector as radius times the angle dθ in radians, we may write:

$$d\boldsymbol{e}_r = \boldsymbol{e}_\theta d\theta \qquad d\boldsymbol{e}_\theta = -\boldsymbol{e}_r d\theta$$

$$d\boldsymbol{e}_r/dt = \boldsymbol{e}_\theta d\theta/dt \qquad d\boldsymbol{e}_\theta/dt = -\boldsymbol{e}_r d\theta/dt$$

$$\boxed{\dot{\boldsymbol{e}}_r = \dot{\theta}\boldsymbol{e}_\theta \qquad \dot{\boldsymbol{e}}_\theta = -\dot{\theta}\boldsymbol{e}_r}$$



Polar coordinates continued ..

- We now differentiate $\boldsymbol{r} = r\boldsymbol{e}_r$ giving $\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{r}\boldsymbol{e}_r + r\dot{\boldsymbol{e}}_r$
- Substituting $\dot{\boldsymbol{e}}_r = \dot{\theta}\boldsymbol{e}_\theta$ reduces to:

$$\boxed{\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta}$$

- where

$$v_r = \dot{r}$$

$$v_\theta = r\dot{\theta}$$

$$v = \sqrt{{v_r}^2 + {v_\theta}^2}$$

Polar coordinates continued ..

- We now differentiate the expression for v ($\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta$) to obtain the acceleration

- thus

$$\boxed{\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{e}_\theta}$$

- Where

$$a_r = \ddot{r} - r\dot{\theta}^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

$$a = \sqrt{{a_r}^2 + {a_\theta}^2}$$

- We may write

$$a_\theta = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

particularly useful in angular momentum of particles

Polar coordinates continued ..

- Magnitude change of Vr

- Direction change of Vr

- Magnitude change of Vθ

- Direction change of Vθ

Polar coordinates continued ..

- Sample problem 2/9
- Sample problem 2/10

2.7 Space Curvilinear Motion

- The general case of three-dimensional motion of a particle

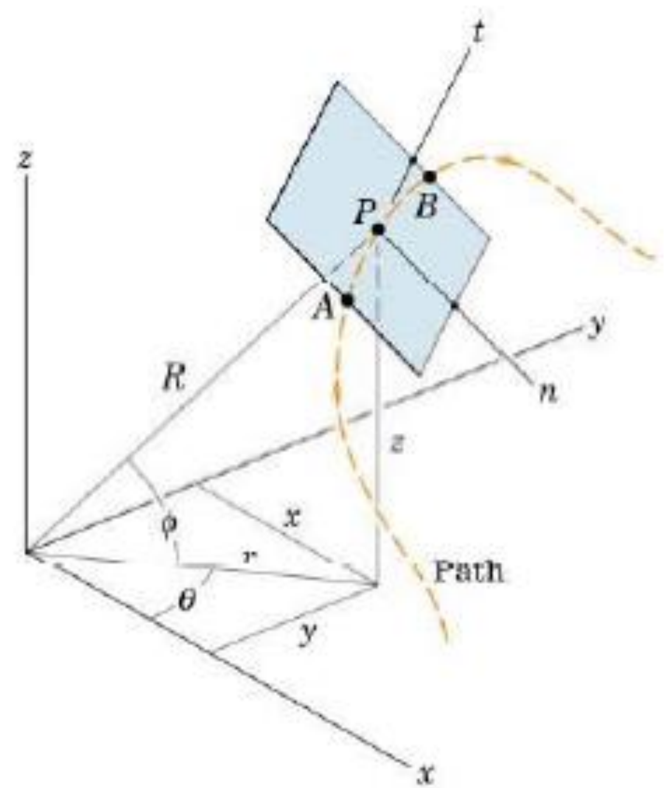


Figure 2/1

RECTANGULAR COORDINATES

- The extension from plane curvilinear motion offers no particular difficulty

$$\boldsymbol{R} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

$$\boldsymbol{v} = \dot{\boldsymbol{R}} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k}$$

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{R}} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} + \ddot{z}\boldsymbol{k}$$

- Note that r is replaced by R

CYLINDRICAL COORDINATES

- From polar coordinates description we extend to cylindrical coordinates with no particular difficulty

$$\boldsymbol{R} = r\boldsymbol{e}_r + z\boldsymbol{k}$$

$$\boldsymbol{v} = \dot{\boldsymbol{R}} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta + \dot{z}\boldsymbol{k}$$

where

$$v_r = \dot{r}$$

$$v_\theta = r\dot{\theta}$$

$$v_z = \dot{z}$$

- Acceleration is written by adding the z-component from the polar equation

$$\boxed{\boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{R}} = \left(\ddot{r} - r\dot{\theta}^2\right)\boldsymbol{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\boldsymbol{e}_\theta + \ddot{z}\boldsymbol{k}}$$

where

$$a_r = \ddot{r} - r\dot{\theta}^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)$$

$$a_z = \ddot{z}$$

$$a = \sqrt{{a_r}^2 + {a_\theta}^2 + {a_z}^2}$$

k has no time derivative

SPHERICAL COORDINATES

- A radial distance and two angles are used to specify the position.
- This is the case in radar measurements

- We designate unit vectors $\boldsymbol{e}_R$, $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\varphi$
- Resulting expressions are:

$$\boxed{\boldsymbol{v} = v_R \boldsymbol{e}_R + v_\theta \boldsymbol{e}_\theta + v_\varphi \boldsymbol{e}_\varphi}$$

where

$$v_R = \dot{R}$$

$$v_\theta = R\dot{\theta}\cos\varphi$$

$$v_\varphi = R\dot{\varphi}$$

SPHERICAL COORDINATES

- Acceleration is:

$$\boxed{\boldsymbol{a} = a_R \boldsymbol{e}_R + a_\theta \boldsymbol{e}_\theta + a_\varphi \boldsymbol{e}_\varphi}$$

where

$$a_R = \ddot{R} - R\dot{\varphi}^2 - R\dot{\theta}^2 cos^2\varphi$$

$$a_\theta = \frac{cos\varphi}{R}\frac{d}{dt}\left(R^2\dot{\theta}\right) - 2R\dot{\theta}\dot{\varphi}sin\varphi$$

$$a_\varphi = \frac{1}{R}\frac{d}{dt}(R^2\dot{\varphi}) + R\dot{\theta}^2 sin\varphi cos\varphi$$

KINEMATICS – SPACE CURVILINEAR, QUESTIONS

Sample problem 2/11 solve

References

- J.L. Meriam and L. G. Krage, Dynamics 2nd edition /6th Edition
- G. Hibberler, 12th Edition