MEC3705 – DYNAMICS

PART I

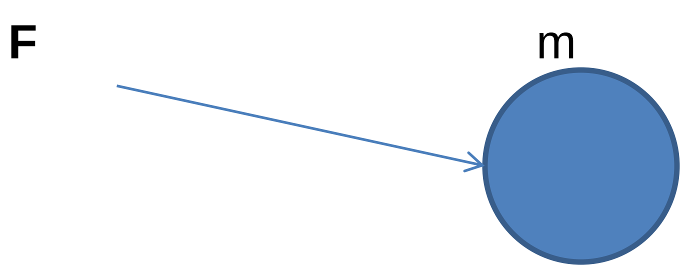
DYNAMICS OF PARTICLES

WORK AND ENERGY

SECTION B. WORK AND ENERGY

3/6 Work and Kinetic Energy

There are two general classes of problems in which the cumulative effects of unbalanced forces acting on a particle are of interest to us. These cases involve (1) integration of the forces with respect to the displacement of the particle and (2) integration of the forces with respect to the time they are applied. We may incorporate the results of these integrations directly into the governing equations of motion so that it becomes unnecessary to solve directly for the acceleration. Integration with respect to displacement leads to the equations of work and energy, which are the subject of this article. Integration with respect to time leads to the equations of impulse and momentum, discussed in Section C.



Definition of Work

We now develop the quantitative meaning of the term "work."* Figure 3/2a shows a force $\mathbf{F}$ acting on a particle at A which moves along the path shown. The position vector $\mathbf{r}$ measured from some convenient origin O locates the particle as it passes point A, and $d\mathbf{r}$ is the differential displacement associated with an infinitesimal movement from A to A'. The work done by the force $\mathbf{F}$ during the displacement $d\mathbf{r}$ is defined as

$$dU = \mathbf{F} \cdot d\mathbf{r}$$

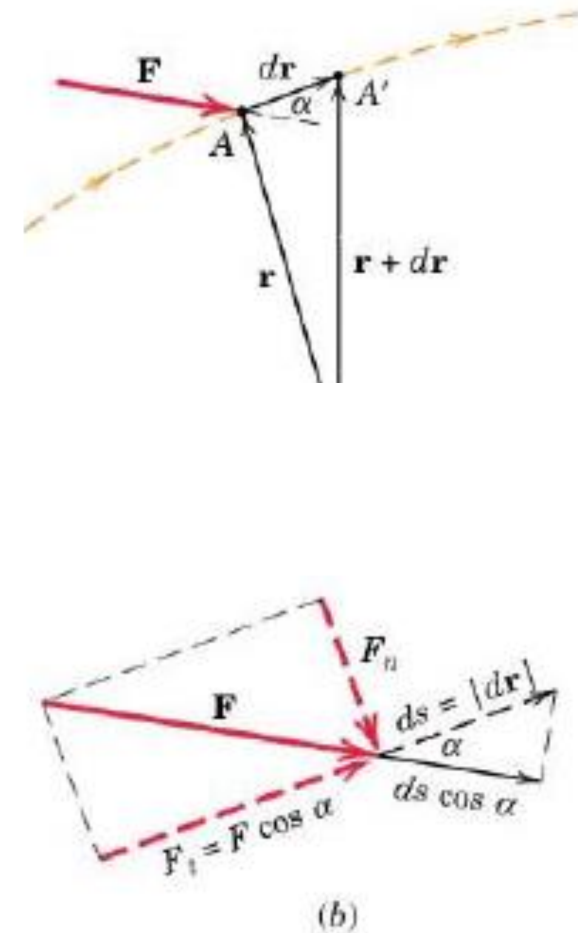


Figure 3/2

- **Work done is equal to dot product of the force vector and the displacement vector**

The magnitude of this dot product is $dU = F\,ds \cos\alpha$, where α is the angle between $\mathbf{F}$ and $d\mathbf{r}$ and where ds is the magnitude of $d\mathbf{r}$. This expression may be interpreted as the displacement multiplied by the force component $F_t = F \cos\alpha$ in the direction of the displacement, as represented by the dashed lines in Fig. 3/2*b*. Alternatively, the work dU may be interpreted as the force multiplied by the displacement component $ds \cos\alpha$ in the direction of the force, as represented by the full lines in Fig. 3/2*b*.

With this definition of work, it should be noted that the component $F_n = F \sin\alpha$ normal to the displacement does no work. Thus, the work dU may be written as

$$dU = F_t\,ds$$

Work is positive if the working component F_t is in the direction of the displacement and negative if it is in the opposite direction.

▪

Calculation of Work

During a finite movement of the point of application of a force, the force does an amount of work equal to

$$U = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \int_1^2 (F_x\,dx + F_y\,dy + F_z\,dz)$$

or

$$U = \int_{s_1}^{s_2} F_t\,ds$$

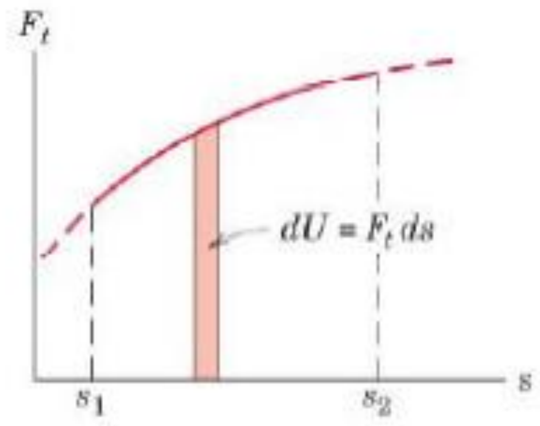


Figure 3/3

Examples of Work

When work must be calculated, we may always begin with the definition of work, $U = \int \mathbf{F} \cdot d\mathbf{r}$, insert appropriate vector expressions for the force $\mathbf{F}$ and the differential displacement vector $d\mathbf{r}$, and carry out the required integration. With some experience, simple work calculations, such as those associated with constant forces, may be performed by inspection. We now formally compute the work associated with three frequently occurring forces: constant forces, spring forces, and weights.

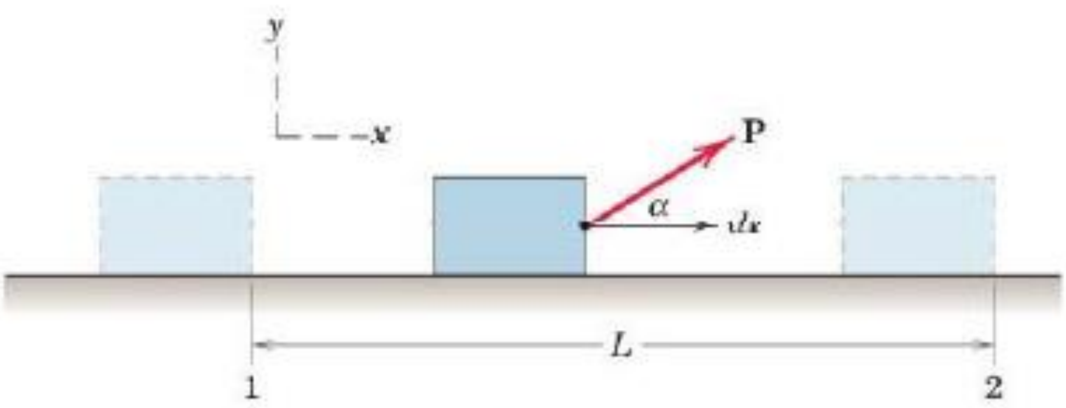


Figure 3/4

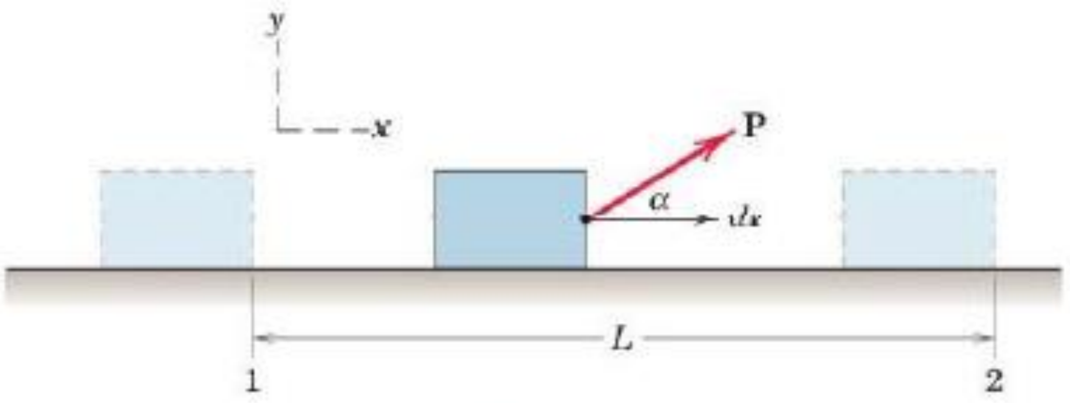


Figure 3/4

(1) Work Associated with a Constant External Force. Consider the constant force **P** applied to the body as it moves from position 1 to position 2, Fig. 3/4. With the force **P** and the differential displacement $d\mathbf{r}$ written as vectors, the work done on the body by the force is

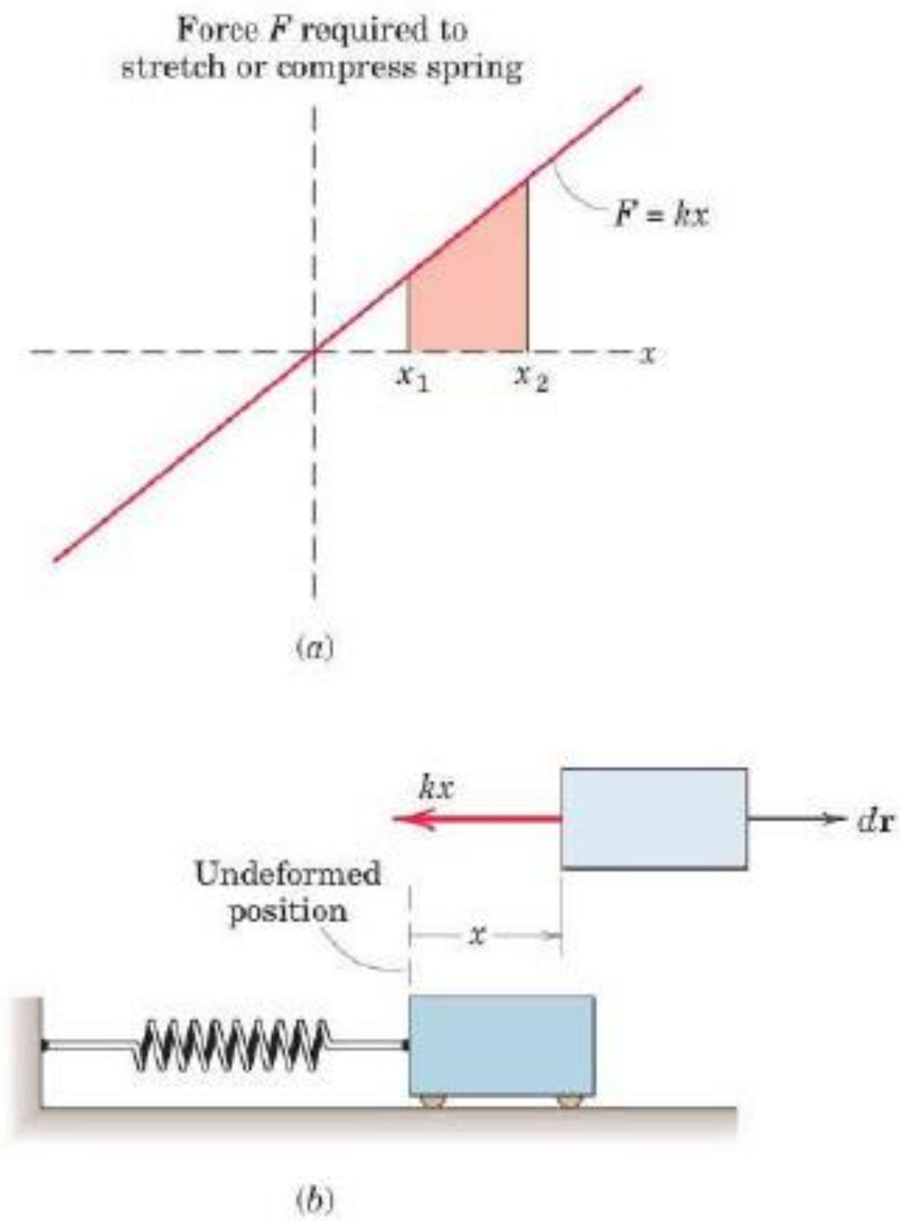
$$U_{1\text{-}2} = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \int_1^2 [(P\cos\alpha)\mathbf{i} + (P\sin\alpha)\mathbf{j}] \cdot dx\,\mathbf{i}$$

$$= \int_{x_1}^{x_2} P\cos\alpha\,dx = P\cos\alpha(x_2 - x_1) = PL\cos\alpha \qquad \textbf{(3/9)}$$

As previously discussed, this work expression may be interpreted as the force component $P\cos\alpha$ times the distance L traveled. Should α be between 90° and 270°, the work would be negative. The force component $P\sin\alpha$ normal to the displacement does no work.

(2) Work Associated with a Spring Force. We consider here the common linear spring of stiffness k where the force required to stretch or compress the spring is proportional to the deformation x, as shown in Fig. 3/5a. We wish to determine the work done on the body by the spring force as the body undergoes an arbitrary displacement from an initial position x_1 to a final position x_2. The force exerted by the spring on the body is $\mathbf{F} = -kx\mathbf{i}$, as shown in Fig. 3/5$b$. From the definition of work, we have

$$U_{1\text{-}2} = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = \int_1^2 (-kx\mathbf{i})\cdot dx\,\mathbf{i} = -\int_{x_1}^{x_2} kx\,dx = \frac{1}{2}k(x_1{}^2 - x_2{}^2) \qquad \textbf{(3/10)}$$



Force F required to stretch or compress spring
F = kx
x
x1
x2
(a)
kx
dr
Undeformed position
x
(b)

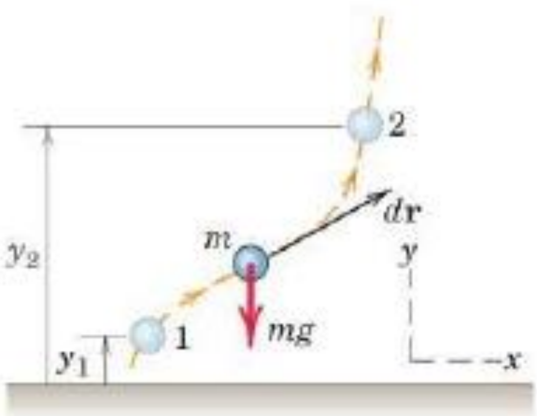
Figure 3/5

The expression $F = kx$ is actually a static relationship which is true only when elements of the spring have no acceleration. The dynamic behavior of a spring when its mass is accounted for is a fairly complex problem which will not be treated here. We shall assume that the mass of the spring is small compared with the masses of other accelerating parts of the system, in which case the linear static relationship will not involve appreciable error.

(3) Work Associated with Weight. *Case (a) g = constant.* If the altitude variation is sufficiently small so that the acceleration of gravity g may be considered constant, the work done by the weight mg of the body shown in Fig. 3/6a as the body is displaced from an arbitrary altitude y_1 to a final altitude y_2 is

$$U_{1\text{-}2} = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = \int_1^2 (-mg\mathbf{j})\cdot(dx\mathbf{i} + dy\mathbf{j})$$

$$= -mg\int_{y_1}^{y_2} dy = -mg(y_2 - y_1) \qquad \textbf{(3/11)}$$



(a)

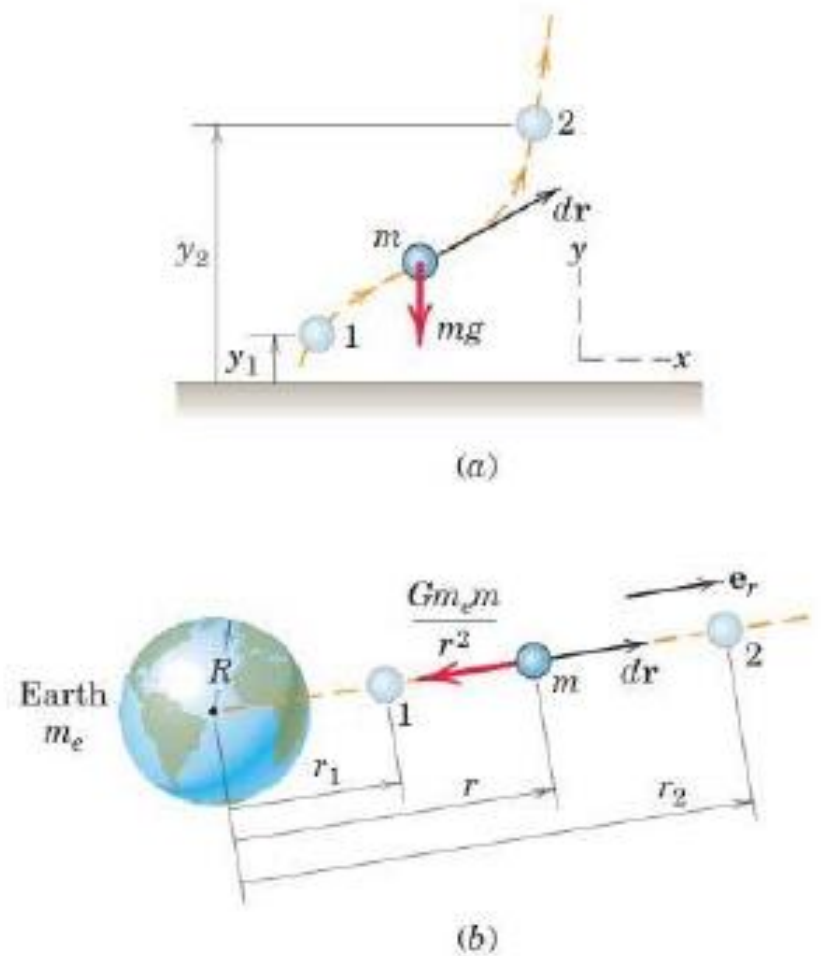

2
$d\mathbf{r}$
m
y_2
y
1
mg
y_1
x
(a)
$\mathbf{e}_r$
$\frac{Gm_em}{r^2}$
Earth
m_e
R
1
m
$d\mathbf{r}$
2
r_1
r
r_2
(b)

Figure 3/6

We see that horizontal movement does not contribute to this work. We also note that if the body rises (perhaps due to other forces not shown), then $(y_2 - y_1) > 0$ and this work is negative. If the body falls, $(y_2 - y_1) < 0$ and the work is positive.

Case (b) $g \neq$ constant. If large changes in altitude occur, then the weight (gravitational force) is no longer constant. We must therefore use the gravitational law (Eq. 1/2) and express the weight as a variable force of magnitude $F = \frac{Gm_e m}{r^2}$, as indicated in Fig. 3/6*b*. Using the radial coordinate shown in the figure allows the work to be expressed as

$$U_{1\text{-}2} = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = \int_1^2 \frac{-Gm_e m}{r^2}\mathbf{e}_r \cdot dr\,\mathbf{e}_r = -Gm_e m\int_{r_1}^{r_2}\frac{dr}{r^2}$$

$$= Gm_e m\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = mgR^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \qquad \textbf{(3/12)}$$

Work and Curvilinear Motion

We now consider the work done on a particle of mass m, Fig. 3/7, moving along a curved path under the action of the force $\mathbf{F}$, which stands for the resultant $\Sigma\mathbf{F}$ of all forces acting on the particle. The position of m is specified by the position vector $\mathbf{r}$, and its displacement along its path during the time dt is represented by the change $d\mathbf{r}$ in its position vector. The work done by $\mathbf{F}$ during a finite movement of the particle from point 1 to point 2 is

$$U_{1\text{-}2} = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = \int_{s_1}^{s_2} F_t\, ds$$

where the limits specify the initial and final end points of the motion.

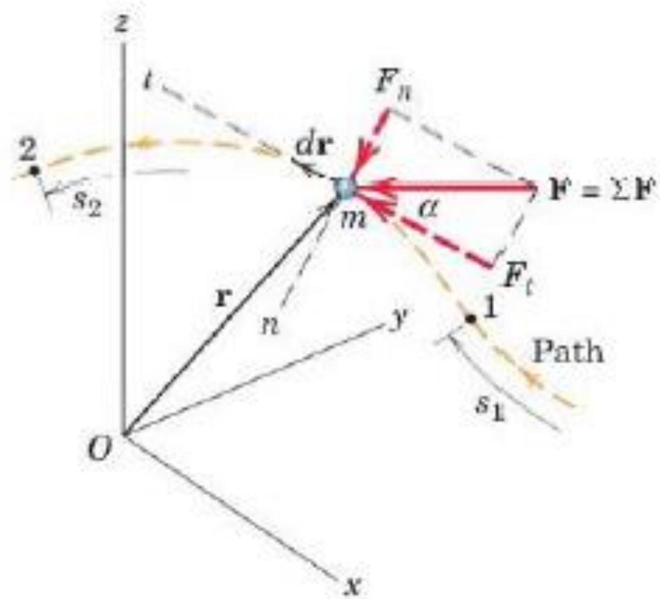


Figure 3/7

When we substitute Newton's second law $\mathbf{F} = m\mathbf{a}$, the expression for the work of all forces becomes

$$U_{1\text{-}2} = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = \int_1^2 m\mathbf{a}\cdot d\mathbf{r}$$

But $\mathbf{a}\cdot d\mathbf{r} = a_t\, ds$, where a_t is the tangential component of the acceleration of m. In terms of the velocity v of the particle, Eq. 2/3 gives $a_t\, ds = v\, dv$. Thus, the expression for the work of $\mathbf{F}$ becomes

$$U_{1\text{-}2} = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = \int_{v_1}^{v_2} mv\, dv = \tfrac{1}{2}m(v_2{}^2 - v_1{}^2) \qquad \textbf{(3/13)}$$

where the integration is carried out between points 1 and 2 along the curve, at which points the velocities have the magnitudes v_1 and v_2, respectively.

Principle of Work and Kinetic Energy

The *kinetic energy* T of the particle is defined as

$$T = \frac{1}{2}mv^2 \qquad \textbf{(3/14)}$$

and is the total work which must be done on the particle to bring it from a state of rest to a velocity v. Kinetic energy T is a scalar quantity with the units of N·m or joules (J) in SI units and ft-lb in U.S. customary units. Kinetic energy is *always* positive, regardless of the direction of the velocity.

Equation 3/13 may be restated as

$$U_{1\text{-}2} = T_2 - T_1 = \Delta T \qquad (3/15)$$

which is the *work-energy equation* for a particle. The equation states that the *total work done* by all forces acting on a particle as it moves from point 1 to point 2 equals the corresponding *change in kinetic energy* of the particle. Although T is always positive, the change ΔT may

be positive, negative, or zero. When written in this concise form, Eq. 3/15 tells us that the work always results in a *change* of kinetic energy.

Alternatively, the work-energy relation may be expressed as the initial kinetic energy T_1 plus the work done $U_{1\text{-}2}$ equals the final kinetic energy T_2, or

$$T_1 + U_{1\text{-}2} = T_2 \qquad (3/15a)$$

When written in this form, the terms correspond to the natural sequence of events. Clearly, the two forms 3/15 and 3/15*a* are equivalent.

Advantages of the Work-Energy Method

We now see from Eq. 3/15 that a major advantage of the method of work and energy is that it avoids the necessity of computing the acceleration and leads directly to the velocity changes as functions of the forces which do work. Further, the work-energy equation involves only those forces which do work and thus give rise to changes in the magnitude of the velocities.

We consider now a system of two particles joined together by a connection which is frictionless and incapable of any deformation. The forces in the connection are equal and opposite, and their points of application necessarily have identical displacement components in the direction of the forces. Therefore, the net work done by these internal forces is zero during any movement of the system. Thus, Eq. 3/15 is applicable to the entire system, where $U_{1\text{-}2}$ is the total or net work done on the system by forces external to it and ΔT is the change, $T_2 - T_1$, in the total kinetic energy of the system. The total kinetic energy is the sum of the kinetic energies of both elements of the system. We thus see that another advantage of the work-energy method is that it enables us to analyze a system of particles joined in the manner described without dismembering the system.

Application of the work-energy method requires isolation of the particle or system under consideration. For a single particle you should draw a *free-body diagram* showing all externally applied forces. For a system of particles rigidly connected without springs, draw an *active-force diagram* showing only those external forces which do work (active forces) on the entire system.*

Power

The capacity of a machine is measured by the time rate at which it can do work or deliver energy. The total work or energy output is not a measure of this capacity since a motor, no matter how small, can deliver a large amount of energy if given sufficient time. On the other hand, a large and powerful machine is required to deliver a large amount of energy in a short period of time. Thus, the capacity of a machine is rated by its *power*, which is defined as the *time rate of doing work*.

Efficiency

The ratio of the work done *by* a machine to the work done *on* the machine during the same time interval is called the *mechanical efficiency* e_m of the machine. This definition assumes that the machine operates uniformly so that there is no accumulation or depletion of energy within it. Efficiency is always less than unity since every device operates with some loss of energy and since energy cannot be created within the machine. In mechanical devices which involve moving parts, there will always be some loss of energy due to the negative work of kinetic friction forces. This work is converted to heat energy which, in turn, is dissipated to the surroundings. The mechanical efficiency at any instant of time may be expressed in terms of mechanical power P by

$$e_m = \frac{P_{\text{output}}}{P_{\text{input}}} \qquad \textbf{(3/17)}$$

In addition to energy loss by mechanical friction, there may also be electrical and thermal energy loss, in which case, the *electrical efficiency* e_e and *thermal efficiency* e_t are also involved. The *overall efficiency* e in such instances is

$$e = e_m e_e e_t$$

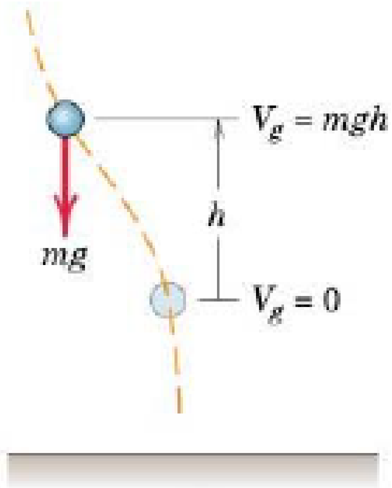
3/7 Potential Energy

In the previous article on work and kinetic energy, we isolated a particle or a combination of joined particles and determined the work done by gravity forces, spring forces, and other externally applied forces acting on the particle or system. We did this to evaluate U in the work-energy equation. In the present article we will introduce the concept of *potential energy* to treat the work done by gravity forces and by spring forces. This concept will simplify the analysis of many problems.

Gravitational Potential Energy

We consider first the motion of a particle of mass m in close proximity to the surface of the earth, where the gravitational attraction (weight) mg is essentially constant, Fig. 3/8a. The *gravitational potential energy* V_g of the particle is defined as the work mgh done *against* the gravitational field to elevate the particle a distance h above some arbitrary reference plane (called a *datum*), where V_g is taken to be zero. Thus, we write the potential energy as

$$V_g = mgh \qquad (3/18)$$

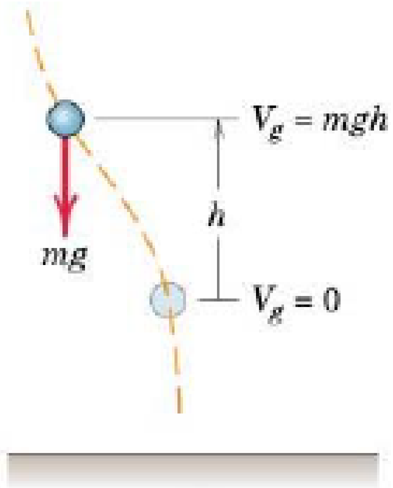


(a)

This work is called potential energy because it may be converted into energy if the particle is allowed to do work on a supporting body while it returns to its lower original datum plane. In going from one level at $h = h_1$ to a higher level at $h = h_2$, the *change* in potential energy becomes

$$\Delta V_g = mg(h_2 - h_1) = mg\Delta h$$

The corresponding work done *by* the gravitational force on the particle is $-mg\Delta h$. Thus, the work done by the gravitational force is the negative of the change in potential energy.



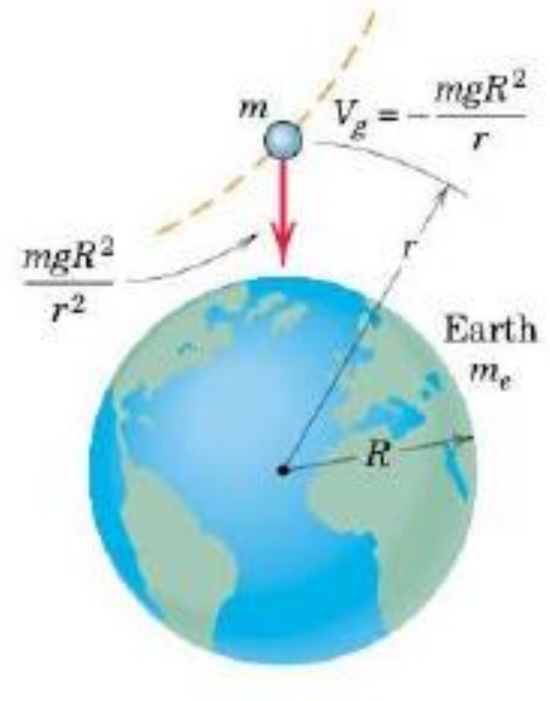
(a)

When large changes in altitude in the field of the earth are encountered, Fig. 3/8*b*, the gravitational force $Gmm_e/r^2 = mgR^2/r^2$ is no longer constant. The work done *against* this force to change the radial position of the particle from r_1 to r_2 is the change $(V_g)_2 - (V_g)_1$ in gravitational potential energy, which is

$$\int_{r_1}^{r_2} mgR^2 \frac{dr}{r^2} = mgR^2\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = (V_g)_2 - (V_g)_1$$

It is customary to take $(V_g)_2 = 0$ when $r_2 = \infty$, so that with this datum we have

$$V_g = -\frac{mgR^2}{r} \qquad \textbf{(3/19)}$$



(b)

Figure 3/8

In going from r_1 to r_2, the corresponding change in potential energy is

$$\Delta V_g = mgR^2 \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$$

which, again, is the *negative* of the work done *by* the gravitational force. We note that the potential energy of a given particle depends only on its position, h or r, and not on the particular path it followed in reaching that position.

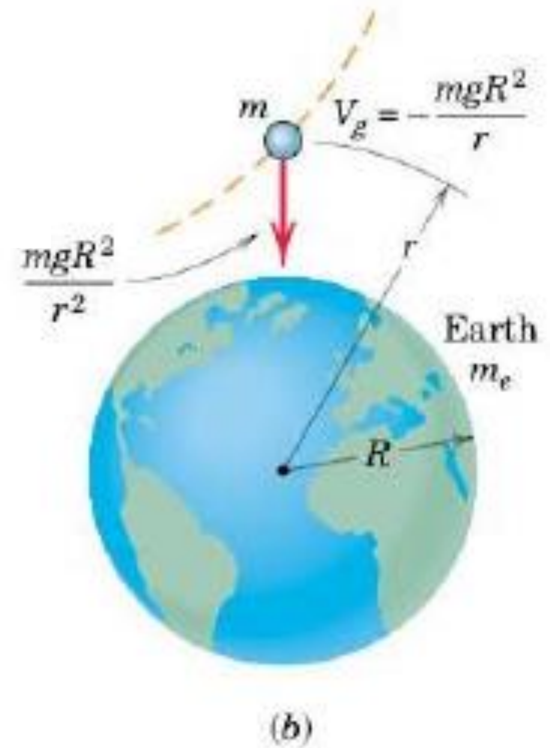


Figure 3/8

Elastic Potential Energy

The second example of potential energy occurs in the deformation of an elastic body, such as a spring. The work which is done on the spring to deform it is stored in the spring and is called its *elastic potential energy* V_e. This energy is recoverable in the form of work done by the spring on the body attached to its movable end during the release of the deformation of the spring. For the one-dimensional linear spring of stiffness k, which we discussed in Art. 3/6 and illustrated in Fig. 3/5, the force supported by the spring at any deformation x, tensile or compressive, from its undeformed position is $F = kx$. Thus, we define the elastic potential energy of the spring as the work done on it to deform it an amount x, and we have

$$V_e = \int_0^x kx\,dx = \frac{1}{2}kx^2 \qquad \textbf{(3/20)}$$

If the deformation, either tensile or compressive, of a spring increases from x_1 to x_2 during the motion, then the change in potential energy of the spring is its final value minus its initial value or

$$\Delta V_e = \frac{1}{2}k({x_2}^2 - {x_1}^2)$$

which is positive. Conversely, if the deformation of a spring decreases during the motion interval, then the change in potential energy of the spring becomes negative. The magnitude of these changes is represented by the shaded trapezoidal area in the F-x diagram of Fig. 3/5a.

Work-Energy Equation

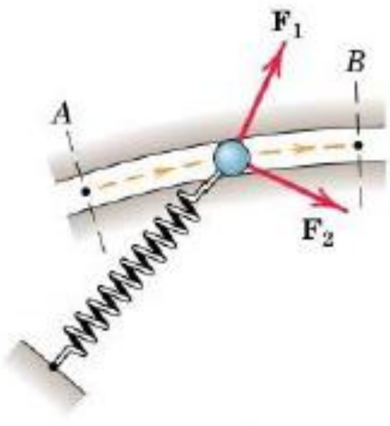
With the elastic member included in the system, we now modify the work-energy equation to account for the potential-energy terms. If $U'_{1\text{-}2}$ stands for the work of all external forces *other than* gravitational forces and spring forces, we may write Eq. 3/15 as $U'_{1\text{-}2} + (-\Delta V_g) + (-\Delta V_e) = \Delta T$ or

$$U'_{1\text{-}2} = \Delta T + \Delta V \qquad \textbf{(3/21)}$$

where ΔV is the change in total potential energy, gravitational plus elastic.

Note that Eq. 3/21 may be rewritten in the equivalent form

$$T_1 + V_1 + U'_{1\text{-}2} = T_2 + V_2 \qquad \mathbf{(3/21}\boldsymbol{a}\mathbf{)}$$



Note that Eq. 3/21 may be rewritten in the equivalent form

$$T_1 + V_1 + U'_{1\text{-}2} = T_2 + V_2 \qquad (3/21a)$$

to B. With the second approach, however, only the initial and final lengths of the spring are required to evaluate ΔV_e. This greatly simplifies the calculation.

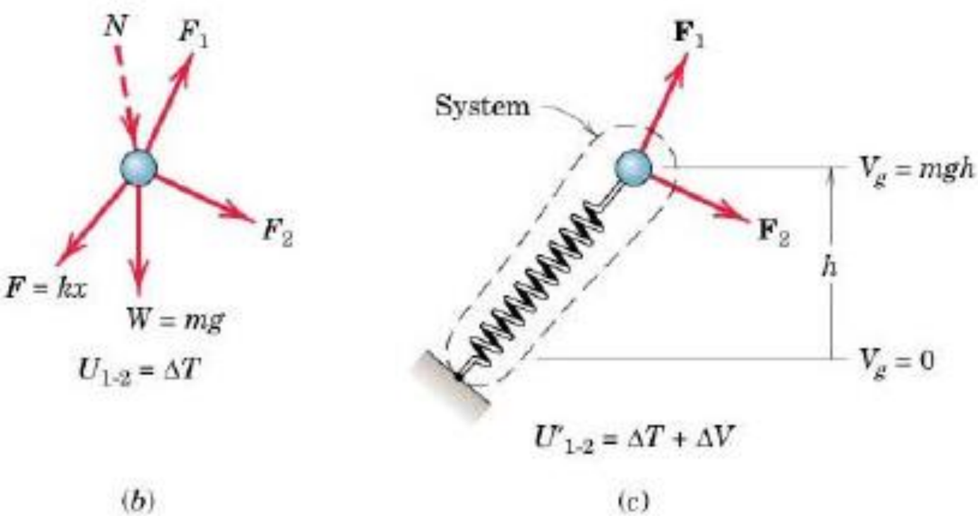


Figure 3/9

For problems where the only forces are gravitational, elastic, and nonworking constraint forces, the U'-term of Eq. 3/21*a* is zero, and the energy equation becomes

$$T_1 + V_1 = T_2 + V_2 \qquad \text{or} \qquad E_1 = E_2 \qquad \textbf{(3/22)}$$

where $E = T + V$ is the total mechanical energy of the particle and its attached spring. When E is constant, we see that transfers of energy between kinetic and potential may take place as long as the total mechanical energy $T + V$ does not change. Equation 3/22 expresses the *law of conservation of dynamical energy*.

Conservative Force Fields*

We have observed that the work done against a gravitational or an elastic force depends only on the net change of position and not on the particular path followed in reaching the new position. Forces with this characteristic are associated with *conservative force fields*, which possess an important mathematical property.

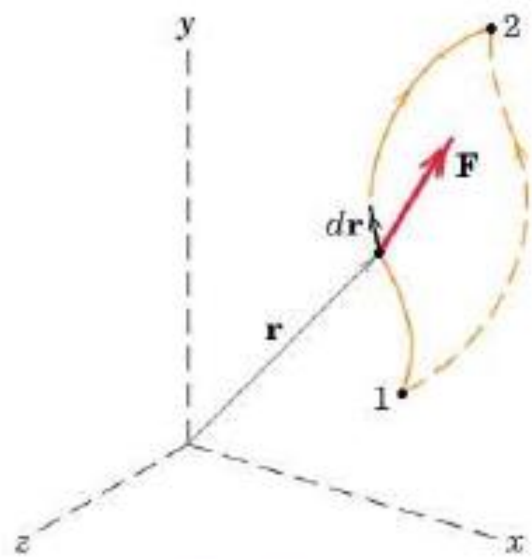


Figure 3/10

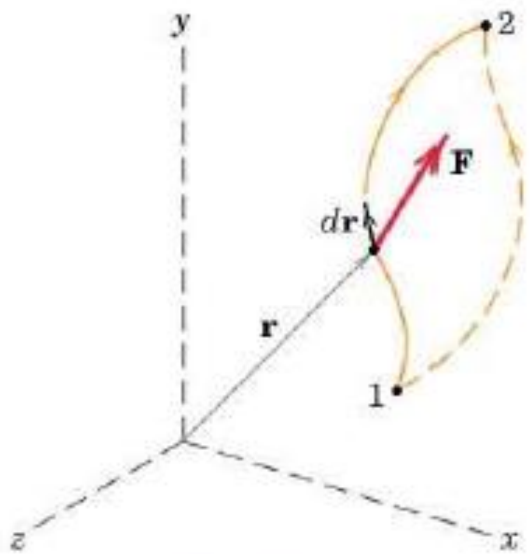


Figure 3/10

Consider a force field where the force $\mathbf{F}$ is a function of the coordinates, Fig. 3/10. The work done by $\mathbf{F}$ during a displacement $d\mathbf{r}$ of its point of application is $dU = \mathbf{F}\cdot d\mathbf{r}$. The total work done along its path from 1 to 2 is

$$U = \int \mathbf{F}\cdot d\mathbf{r} = \int (F_x\,dx + F_y\,dy + F_z\,dz)$$

The integral $\int \mathbf{F}\cdot d\mathbf{r}$ is a line integral which depends, in general, on the particular path followed between any two points 1 and 2 in space. If, however, $\mathbf{F}\cdot d\mathbf{r}$ is an *exact differential*† $-dV$ of some scalar function V of the coordinates, then

$$U_{1\text{-}2} = \int_{V_1}^{V_2} -dV = -(V_2 - V_1) \qquad \textbf{(3/23)}$$

which depends only on the end points of the motion and which is thus *independent* of the path followed. The minus sign before dV is arbitrary but is chosen to agree with the customary designation of the sign of potential energy change in the gravity field of the earth.

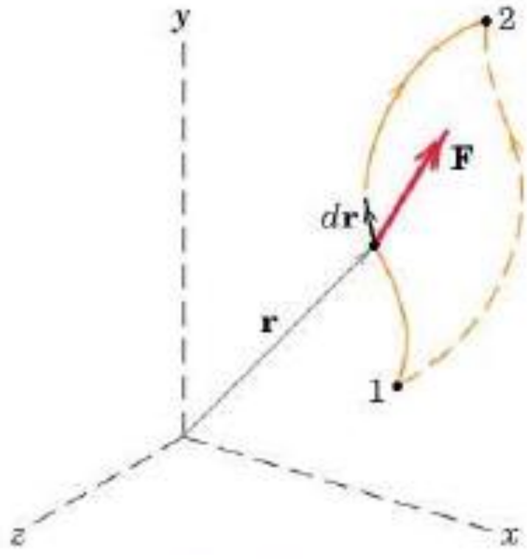


Figure 3/10

If V exists, the differential change in V becomes

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

Comparison with $-dV = \mathbf{F}\cdot d\mathbf{r} = F_x\,dx + F_y\,dy + F_z\,dz$ gives us

$$F_x = -\frac{\partial V}{\partial x} \qquad F_y = -\frac{\partial V}{\partial y} \qquad F_z = -\frac{\partial V}{\partial z}$$

The force may also be written as the vector

$$\mathbf{F} = -\boldsymbol{\nabla} V \tag{3/24}$$

where the symbol $\boldsymbol{\nabla}$ stands for the vector operator "del", which is

$$\boldsymbol{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

The quantity V is known as the *potential function*, and the expression $\boldsymbol{\nabla} V$ is known as the *gradient of the potential function*.

When force components are derivable from a potential as described, the force is said to be *conservative*, and the work done by **F** between any two points is independent of the path followed.

Next is impulse and momentum

References

- J.L. Meriam and L. G. Krage, Dynamics 6[TH] edition