MEC3705 – DYNAMICS

PART II

DYNAMICS OF RIGID BODIES

KINEMATICS OF RIGID BODIES

5 PLANE KINEMATICS OF RIGID BODIES

CHAPTER OUTLINE

Rigid-Body Assumption

In the previous chapter we defined a *rigid body* as a system of particles for which the distances between the particles remain unchanged. Thus, if each particle of such a body is located by a position vector from reference axes attached to and rotating with the body, there will be no change in any position vector as measured from these axes. This is, of course, an ideal case since all solid materials change shape to some extent when forces are applied to them.

Plane Motion

A rigid body executes plane motion when all parts of the body move in parallel planes. For convenience, we generally consider the *plane of motion* to be the plane which contains the mass center, and we treat the body as a thin slab whose motion is confined to the plane of the slab. This idealization adequately describes a very large category of rigid-body motions encountered in engineering.

The plane motion of a rigid body may be divided into several categories, as represented in Fig. 5/1.

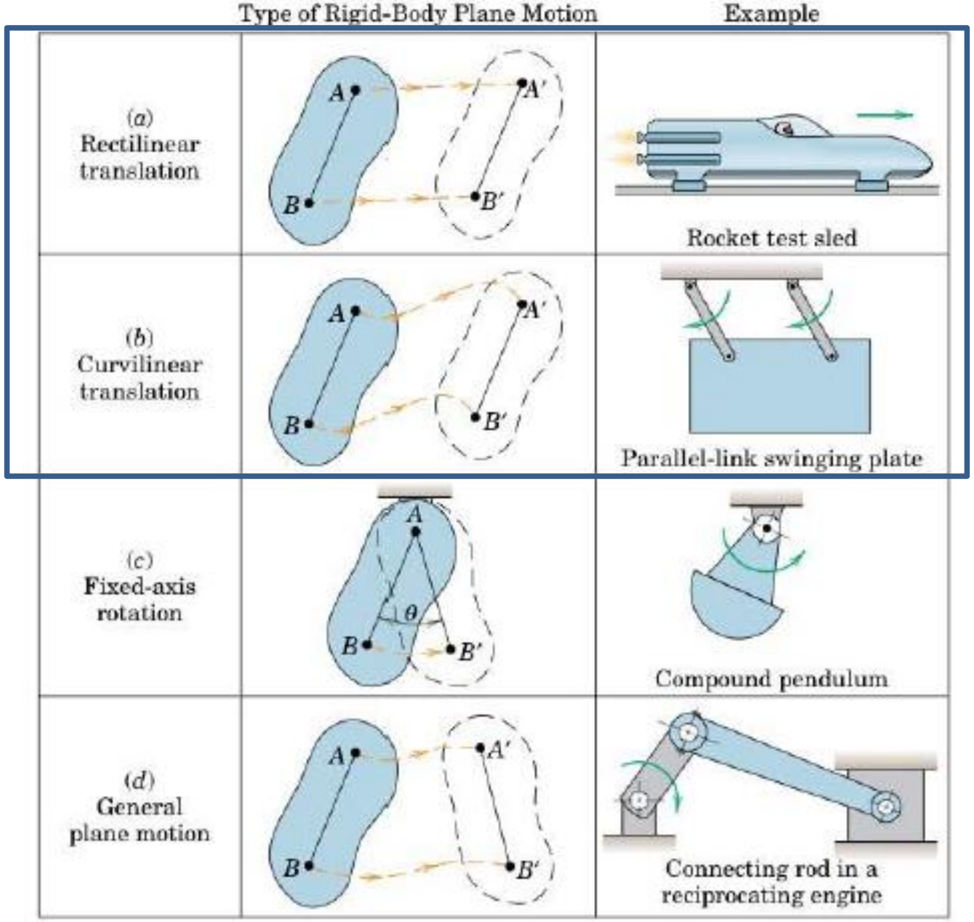


Figure 5/1

Translation is defined as any motion in which every line in the body remains parallel to its original position at all times. In translation there is *no rotation of any line in the body*. In *rectilinear translation*, part a of Fig. 5/1, all points in the body move in parallel straight lines. In *curvilinear translation*, part b, all points move on congruent curves. We note that in each of the two cases of translation, the motion of the body is completely specified by the motion of any point in the body, since all points have the same motion. Thus, our earlier study of the motion of a point (particle) in Chapter 2 enables us to describe completely the translation of a rigid body.

Rotation about a fixed axis, part *c* of Fig. 5/1, is the angular motion about the axis. It follows that all particles in a rigid body move in circular paths about the axis of rotation, and all lines in the body which are perpendicular to the axis of rotation (including those which do not pass through the axis) rotate through the same angle in the same time. Again, our discussion in Chapter 2 on the circular motion of a point enables us to describe the motion of a rotating rigid body, which is treated in the next article.

General plane motion of a rigid body, part *d* of Fig. 5/1, is a combination of translation and rotation. We will utilise the principles of relative motion covered in Art. 2/8 to describe general plane motion.

Note that in each of the examples cited, the actual paths of all particles in the body are projected onto the single plane of motion as represented in each figure.

Analysis of the plane motion of rigid bodies is accomplished either by directly calculating the absolute displacements and their time derivatives from the geometry involved or by utilizing the principles of relative motion. Each method is important and useful and will be covered in turn in the articles which follow.

5/2 Rotation

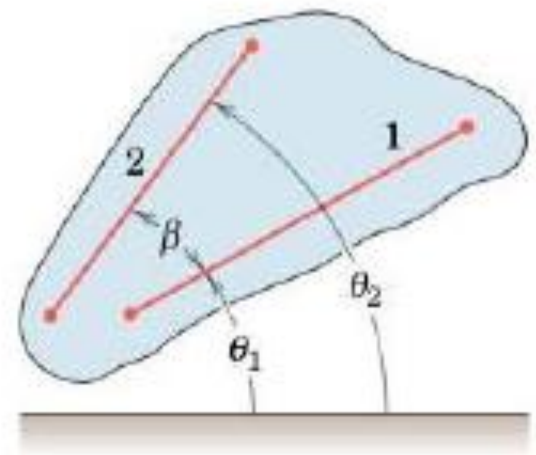


Figure 5/2

The rotation of a rigid body is described by its angular motion. Figure 5/2 shows a rigid body which is rotating as it undergoes plane motion in the plane of the figure. The angular positions of any two lines 1 and 2 attached to the body are specified by θ_1 and θ_2 measured from any convenient fixed reference direction. Because the angle β is invariant, the relation $\theta_2 = \theta_1 + \beta$ upon differentiation with respect to time gives $\dot{\theta}_2 = \dot{\theta}_1$ and $\ddot{\theta}_2 = \ddot{\theta}_1$ or, during a finite interval, $\Delta\theta_2 = \Delta\theta_1$.

Thus, *all lines on a rigid body in its plane of motion have the same angular displacement, the same angular velocity, and the same angular acceleration*.

Angular-Motion Relations

The angular velocity ω and angular acceleration α of a rigid body in plane rotation are, respectively, the first and second time derivatives of the angular position coordinate θ of any line in the plane of motion of the body. These definitions give

$$\omega = \frac{d\theta}{dt} = \dot{\theta}$$

$$\alpha = \frac{d\omega}{dt} = \dot{\omega} \qquad \text{or} \qquad \alpha = \frac{d^2\theta}{dt^2} = \ddot{\theta}$$

$$\omega\, d\omega = \alpha\, d\theta \qquad \text{or} \qquad \dot{\theta}\, d\dot{\theta} = \ddot{\theta}\, d\theta$$

For rotation with *constant* angular acceleration, the integrals of Eqs. 5/1 becomes

$$\omega = \omega_0 + \alpha t$$

$$\omega^2 = {\omega_0}^2 + 2\alpha(\theta - \theta_0)$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

Here θ_0 and ω_0 are the values of the angular position coordinate and angular velocity, respectively, at $t = 0$, and t is the duration of the motion considered. You should be able to carry out these integrations easily, as they are completely analogous to the corresponding equations for rectilinear motion with constant acceleration covered in Art. 2/2.

Rotation about a Fixed Axis

When a rigid body rotates about a fixed axis, all points other than those on the axis move in concentric circles about the fixed axis. Thus, for the rigid body in Fig. 5/3 rotating about a fixed axis normal to the plane of the figure through O, any point such as A moves in a circle of radius r. From the previous discussion in Art. 2/5, you should already be familiar with the relationships between the linear motion of A and the angular motion of the line normal to its path, which is also the angular motion of the rigid body. With the notation $\omega = \dot{\theta}$ and $\alpha = \dot{\omega} = \ddot{\theta}$ for the angular velocity and angular acceleration, respectively, of the body we have Eqs. 2/11, rewritten as

$$\begin{aligned} v &= r\omega \\ a_n &= r\omega^2 = v^2/r = v\omega \\ a_t &= r\alpha \end{aligned} \qquad (5/2)$$

Rotation about a Fixed Axis

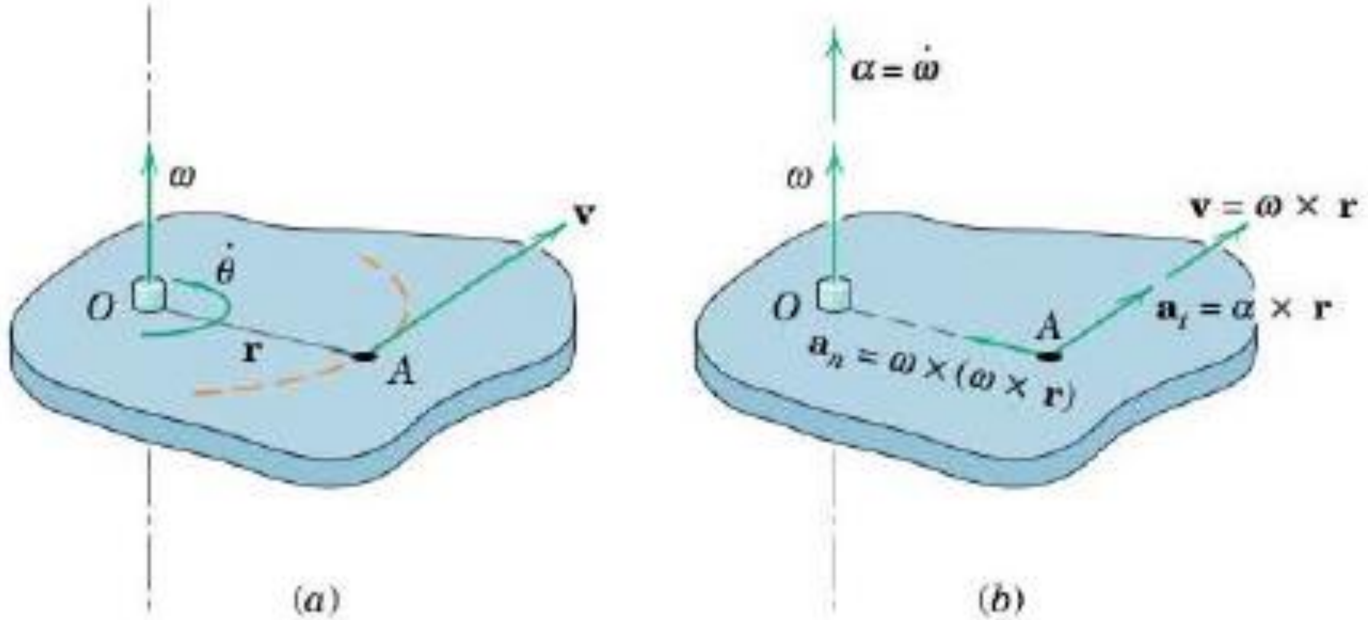


Figure 5/4

These quantities may be expressed alternatively using the cross-product relationship of vector notation. The vector formulation is especially important in the analysis of three-dimensional motion. The angular velocity of the rotating body may be expressed by the vector $\boldsymbol{\omega}$ normal to the plane of rotation and having a sense governed by the right hand rule, as shown in Fig. 5/4a. From the definition of the vector cross product, we see that the vector $\mathbf{v}$ is obtained by crossing $\boldsymbol{\omega}$ into $\mathbf{r}$. This cross product gives the correct magnitude and direction for $\mathbf{v}$ and we write

$$\mathbf{v} = \dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$$

The order of the vectors to be crossed must be retained. The reverse order gives $\mathbf{r} \times \boldsymbol{\omega} = -\mathbf{v}$.

The acceleration of point A is obtained by differentiating the cross-product expression for **v**, which gives

$$\mathbf{a} = \dot{\mathbf{v}} = \boldsymbol{\omega} \times \dot{\mathbf{r}} + \dot{\boldsymbol{\omega}} \times \mathbf{r}$$
$$= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \dot{\boldsymbol{\omega}} \times \mathbf{r}$$
$$= \boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\alpha} \times \mathbf{r}$$

Here $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$ stands for the angular acceleration of the body. Thus, the vector equivalents to Eqs. 5/2 are

$$\begin{aligned} \mathbf{v} &= \boldsymbol{\omega} \times \mathbf{r} \\ \mathbf{a}_n &= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \\ \mathbf{a}_t &= \boldsymbol{\alpha} \times \mathbf{r} \end{aligned} \qquad \textbf{(5/3)}$$

and are shown in Fig. 5/4*b*.

For three-dimensional motion of a rigid body, the angular-velocity vector $\boldsymbol{\omega}$ may change direction as well as magnitude, and in this case, the angular acceleration, which is the time derivative of angular velocity, $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$, will no longer be in the same direction as $\boldsymbol{\omega}$.

Sample Problem 5/1

A flywheel rotating freely at 1800 rev/min clockwise is subjected to a variable counterclockwise torque which is first applied at time $t = 0$. The torque produces a counterclockwise angular acceleration $\alpha = 4t$ rad/s^2, where t is the time in seconds during which the torque is applied. Determine (*a*) the time required for the flywheel to reduce its clockwise angular speed to 900 rev/min, (*b*) the time required for the flywheel to reverse its direction of rotation, and (*c*) the total number of revolutions, clockwise plus counterclockwise, turned by the flywheel during the first 14 seconds of torque application.

Solution. The counterclockwise direction will be taken arbitrarily as positive.

(a) Since α is a known function of the time, we may integrate it to obtain angular velocity. With the initial angular velocity of $-1800(2\pi)/60 = -60\pi$ rad/s, we have

$$[d\omega = \alpha\, dt] \qquad \int_{-60\pi}^{\omega} d\omega = \int_0^t 4t\, dt \qquad \omega = -60\pi + 2t^2$$

Substituting the clockwise angular speed of 900 rev/min or $\omega = -900(2\pi)/60 = -30\pi$ rad/s gives

$$-30\pi = -60\pi + 2t^2 \qquad t^2 = 15\pi \qquad t = 6.86 \text{ s} \qquad \textit{Ans.}$$

(b) The flywheel changes direction when its angular velocity is momentarily zero. Thus,

$$0 = -60\pi + 2t^2 \qquad t^2 = 30\pi \qquad t = 9.71 \text{ s} \qquad \textit{Ans.}$$

(c) The total number of revolutions through which the flywheel turns during 14 seconds is the number of clockwise turns N_1 during the first 9.71 seconds, plus the number of counterclockwise turns N_2 during the remainder of the interval. Integrating the expression for ω in terms of t gives us the angular displacement in radians. Thus, for the first interval

$[d\theta = \omega\, dt]$

$$\int_0^{\theta_1} d\theta = \int_0^{9.71} (-60\pi + 2t^2)\, dt$$

$$\theta_1 = \left[-60\pi t + \frac{2}{3}t^3\right]_0^{9.71} = -1220 \text{ rad}$$

or $N_1 = 1220/2\pi = 194.2$ revolutions clockwise.

For the second interval

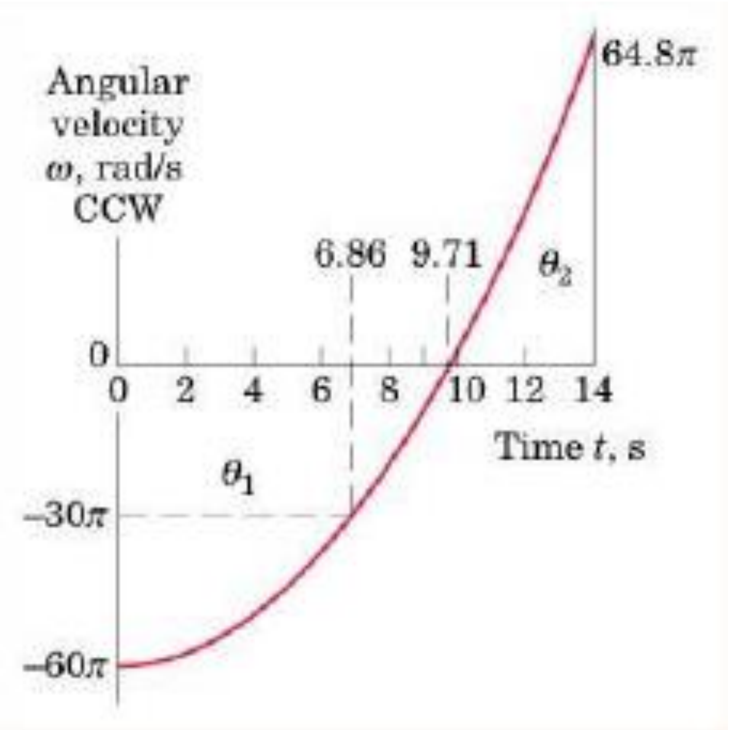
$$\int_0^{\theta_2} d\theta = \int_{9.71}^{14} (-60\pi + 2t^2)\, dt$$

$$\theta_2 = \left[-60\pi t + \frac{2}{3}t^3\right]_{9.71}^{14} = 410 \text{ rad}$$

or $N_2 = 410/2\pi = 65.3$ revolutions counterclockwise. Thus, the total number of revolutions turned during the 14 seconds is

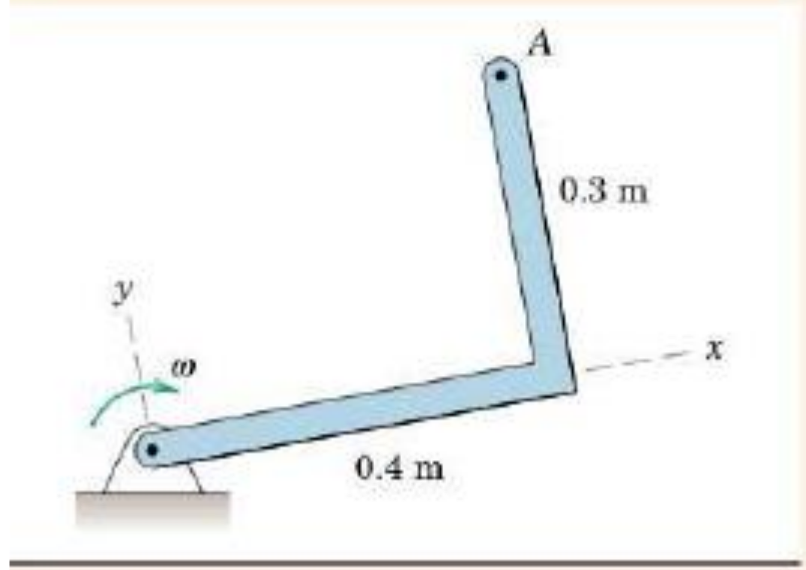
$$N = N_1 + N_2 = 194.2 + 65.3 = 259 \text{ rev} \qquad \textit{Ans.}$$

We have plotted ω versus t and we see that θ_1 is represented by the negative area and θ_2 by the positive area. If we had integrated over the entire interval in one step, we would have obtained $|\theta_2| - |\theta_1|$.


Angular velocity ω, rad/s CCW
64.8π
6.86
9.71
θ2
0
0 2 4 6 8 10 12 14
Time t, s
θ1
–30π
–60π

Sample Problem 5/3

The right-angle bar rotates clockwise with an angular velocity which is decreasing at the rate of 4 rad/s^2. Write the vector expressions for the velocity and acceleration of point A when $\omega = 2$ rad/s.



Solution. Using the right-hand rule gives

$$\boldsymbol{\omega} = -2\mathbf{k} \text{ rad/s} \qquad \text{and} \qquad \boldsymbol{\alpha} = +4\mathbf{k} \text{ rad/s}^2$$

The velocity and acceleration of A become

$[\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}]$ $\qquad \mathbf{v} = -2\mathbf{k} \times (0.4\mathbf{i} + 0.3\mathbf{j}) = 0.6\mathbf{i} - 0.8\mathbf{j}$ m/s *Ans.*

$[\mathbf{a}_n = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})]$ $\qquad \mathbf{a}_n = -2\mathbf{k} \times (0.6\mathbf{i} - 0.8\mathbf{j}) = -1.6\mathbf{i} - 1.2\mathbf{j}$ m/s²

$[\mathbf{a}_t = \boldsymbol{\alpha} \times \mathbf{r}]$ $\qquad \mathbf{a}_t = 4\mathbf{k} \times (0.4\mathbf{i} + 0.3\mathbf{j}) = -1.2\mathbf{i} + 1.6\mathbf{j}$ m/s²

$[\mathbf{a} = \mathbf{a}_n + \mathbf{a}_t]$ $\qquad \mathbf{a} = -2.8\mathbf{i} + 0.4\mathbf{j}$ m/s² *Ans.*

The magnitudes of **v** and **a** are

$$v = \sqrt{0.6^2 + 0.8^2} = 1 \text{ m/s} \qquad \text{and} \qquad a = \sqrt{2.8^2 + 0.4^2} = 2.83 \text{ m/s}^2$$

5/3 Absolute Motion

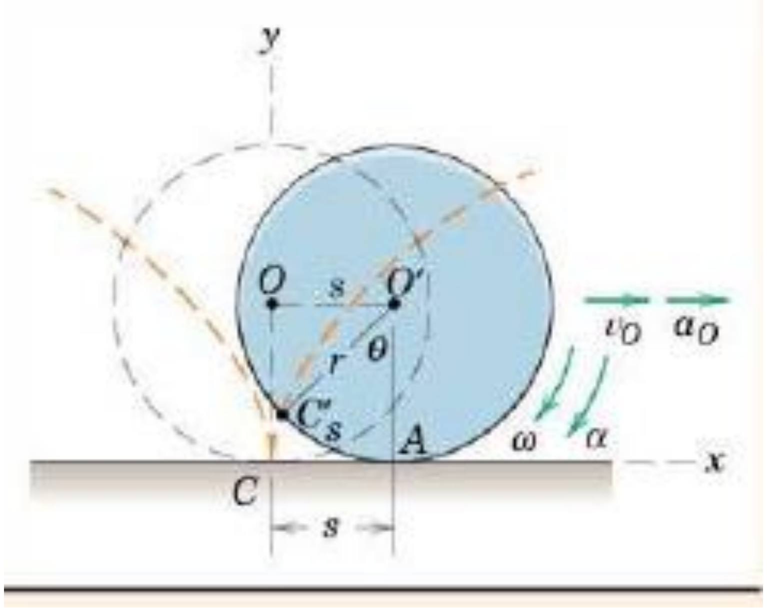
We now develop the approach of absolute-motion analysis to describe the plane kinematics of rigid bodies. In this approach, we make use of the geometric relations which define the configuration of the body involved and then proceed to take the time derivatives of the defining geometric relations to obtain velocities and accelerations.

The absolute-motion approach to rigid-body kinematics is quite straightforward, provided the configuration lends itself to a geometric description which is not overly complex. If the geometric configuration is awkward or complex, analysis by the principles of relative motion may be preferable. Relative-motion analysis is treated in this chapter beginning with Art. 5/4. The choice between absolute- and relative-motion analyses is best made after experience has been gained with both approaches.

The next three sample problems illustrate the application of absolute-motion analysis to three commonly encountered situations. The kinematics of a rolling wheel, treated in Sample Problem 5/4, is especially important and will be useful in much of the problem work because the rolling wheel in various forms is such a common element in mechanical systems.

Sample Problem 5/4

A wheel of radius r rolls on a flat surface without slipping. Determine the angular motion of the wheel in terms of the linear motion of its center O. Also determine the acceleration of a point on the rim of the wheel as the point comes into contact with the surface on which the wheel rolls.



Solution. The figure shows the wheel rolling to the right from the dashed to the full position without slipping. The linear displacement of the center O is s, which is also the arc length $C'A$ along the rim on which the wheel rolls. The radial line CO rotates to the new position $C'O'$ through the angle θ, where θ is measured from the vertical direction. If the wheel does not slip, the arc $C'A$ must equal the distance s. Thus, the displacement relationship and its two time derivatives give

$$s = r\theta$$

$$v_O = r\omega \qquad \textit{Ans.}$$

$$a_O = r\alpha$$

where $v_O = \dot{s}$, $a_O = \dot{v}_O = \ddot{s}$, $\omega = \dot{\theta}$, and $\alpha = \dot{\omega} = \ddot{\theta}$. The angle θ, of course, must be in radians. The acceleration a_O will be directed in the sense opposite to that of v_O if the wheel is slowing down. In this event, the angular acceleration α will have the sense opposite to that of ω.

The origin of fixed coordinates is taken arbitrarily but conveniently at the point of contact between C on the rim of the wheel and the ground. When point C has moved along its cycloidal path to C', its new coordinates and their time derivatives become

$$x = s - r\sin\theta = r(\theta - \sin\theta) \qquad y = r - r\cos\theta = r(1 - \cos\theta)$$

$$\dot{x} = r\dot{\theta}(1 - \cos\theta) = v_O(1 - \cos\theta) \qquad \dot{y} = r\dot{\theta}\sin\theta = v_O\sin\theta$$

$$\ddot{x} = \dot{v}_O(1 - \cos\theta) + v_O\dot{\theta}\sin\theta \qquad \ddot{y} = \dot{v}_O\sin\theta + v_O\dot{\theta}\cos\theta$$

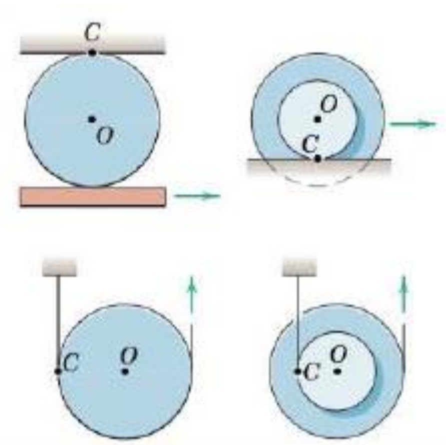
$$= a_O(1 - \cos\theta) + r\omega^2\sin\theta \qquad = a_O\sin\theta + r\omega^2\cos\theta$$

For the desired instant of contact, $\theta = 0$ and

$$\ddot{x} = 0 \qquad \text{and} \qquad \ddot{y} = r\omega^2 \qquad \textit{Ans.}$$

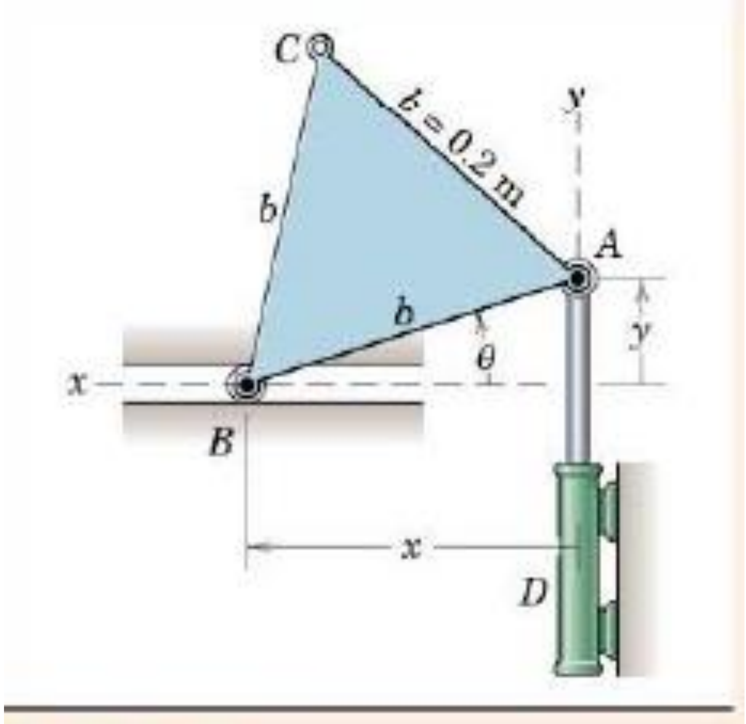
Thus, the acceleration of the point C on the rim at the instant of contact with the ground depends only on r and ω and is directed toward the center of the wheel. If desired, the velocity and acceleration of C at any position θ may be obtained by writing the expressions $\mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}$ and $\mathbf{a} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j}$.

Application of the kinematic relationships for a wheel which rolls without slipping should be recognized for various configurations of rolling wheels such as those illustrated on the right. If a wheel slips as it rolls, the foregoing relations are no longer valid.



Sample Problem 5/6

Motion of the equilateral triangular plate ABC in its plane is controlled by the hydraulic cylinder D. If the piston rod in the cylinder is moving upward at the constant rate of 0.3 m/s during an interval of its motion, calculate for the instant when $\theta = 30°$ the velocity and acceleration of the center of the roller B in the horizontal guide and the angular velocity and angular acceleration of edge CB.



Solution. With the x-y coordinates chosen as shown, the given motion of A is $v_A = \dot{y} = 0.3$ m/s and $a_A = \ddot{y} = 0$. The accompanying motion of B is given by x and its time derivatives, which may be obtained from $x^2 + y^2 = b^2$. Differentiating gives

$$x\dot{x} + y\dot{y} = 0 \qquad \dot{x} = -\frac{y}{x}\dot{y}$$

$$x\ddot{x} + \dot{x}^2 + y\ddot{y} + \dot{y}^2 = 0 \qquad \ddot{x} = -\frac{\dot{x}^2 + \dot{y}^2}{x} - \frac{y}{x}\ddot{y}$$

With $y = b \sin\theta$, $x = b\cos\theta$, and $\ddot{y} = 0$, the expressions become

$$v_B = \dot{x} = -v_A \tan\theta$$

$$a_B = \ddot{x} = -\frac{{v_A}^2}{b}\sec^3\theta$$

Substituting the numerical values $v_A = 0.3$ m/s and $\theta = 30°$ gives

$$v_B = -0.3\left(\frac{1}{\sqrt{3}}\right) = -0.1732 \text{ m/s}$$ *Ans.*

$$a_B = -\frac{(0.3)^2(2/\sqrt{3})^3}{0.2} = -0.693 \text{ m/s}^2$$ *Ans.*

The negative signs indicate that the velocity and acceleration of B are both to the right since x and its derivatives are positive to the left.

The angular motion of CB is the same as that of every line on the plate, including AB. Differentiating $y = b \sin \theta$ gives

$$\dot{y} = b\dot{\theta} \cos \theta \qquad \omega = \dot{\theta} = \frac{v_A}{b} \sec \theta$$

The angular acceleration is

$$\alpha = \dot{\omega} = \frac{v_A}{b} \dot{\theta} \sec \theta \tan \theta = \frac{{v_A}^2}{b^2} \sec^2 \theta \tan \theta$$

Substitution of the numerical values gives

$$\omega = \frac{0.3}{0.2} \frac{2}{\sqrt{3}} = 1.732 \text{ rad/s} \qquad \textit{Ans.}$$

$$\alpha = \frac{(0.3)^2}{(0.2)^2} \left(\frac{2}{\sqrt{3}}\right)^2 \frac{1}{\sqrt{3}} = 1.732 \text{ rad/s}^2 \qquad \textit{Ans.}$$

Both ω and α are counterclockwise since their signs are positive in the sense of the positive measurement of θ.

5/4 RELATIVE VELOCITY

The second approach to rigid-body kinematics is to use the principles of relative motion. In Art. 2/8 we developed these principles for motion relative to translating axes and applied the relative-velocity equation

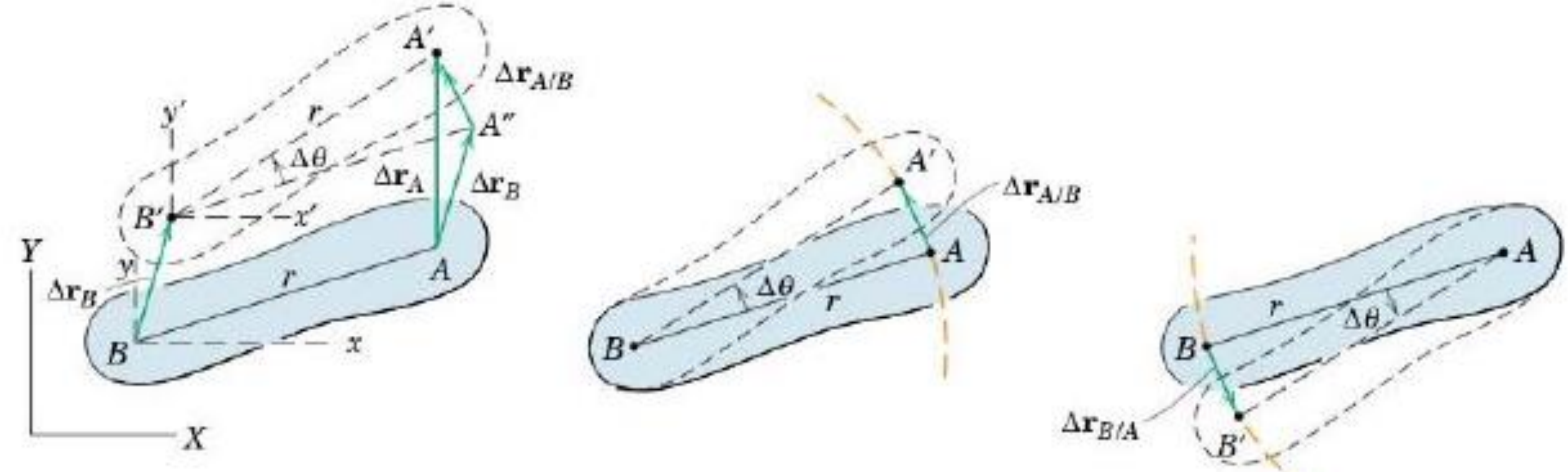
$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B} \qquad \textbf{[2/20]}$$

to the motions of two particles A and B.

Relative Velocity Due to Rotation

We now choose two points on the *same* rigid body for our two particles. The consequence of this choice is that the motion of one point as seen by an observer translating with the other point must be circular since the radial distance to the observed point from the reference point does not change. This observation is the *key* to the successful understanding of a large majority of problems in the plane motion of rigid bodies.

This concept is illustrated in Fig. 5/5a, which shows a rigid body moving in the plane of the figure from position AB to $A'B'$ during time Δt. This movement may be visualized as occurring in two parts. First, the body translates to the parallel position $A''B'$ with the displacement $\Delta \mathbf{r}_B$. Second, the body rotates about B' through the angle $\Delta\theta$. From the nonrotating reference axes x'-y' attached to the reference point B', you can see that this remaining motion of the body is one of simple rotation about B', giving rise to the displacement $\Delta \mathbf{r}_{A/B}$ of A with respect to B. To the nonrotating observer attached to B, the body appears to undergo fixed-axis rotation about B with A executing circular motion as emphasized in Fig. 5/5b. Therefore, the relationships developed for circular motion in Arts. 2/5 and 5/2 and cited as Eqs. 2/11 and 5/2 (or 5/3) describe the relative portion of the motion of point A.



(a)

Motion relative to B
(b)

Motion relative to A
(c)

Figure 5/5

With B as the reference point, we see from Fig. 5/5a that the total displacement of A is

$$\Delta \mathbf{r}_A = \Delta \mathbf{r}_B + \Delta \mathbf{r}_{A/B}$$

where $\Delta \mathbf{r}_{A/B}$ has the magnitude $r\Delta\theta$ as $\Delta\theta$ approaches zero. We note that the *relative linear motion* $\Delta \mathbf{r}_{A/B}$ is accompanied by the *absolute angular motion* $\Delta\theta$, as seen from the translating axes x'-y'. Dividing the expression for $\Delta \mathbf{r}_A$ by the corresponding time interval Δt and passing to the limit, we obtain the relative-velocity equation

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B} \qquad \textbf{(5/4)}$$

This expression is the same as Eq. 2/20, with the one restriction that the distance r between A and B remains constant. The magnitude of the relative velocity is thus seen to be $v_{A/B} = \lim_{\Delta t \to 0} (|\Delta \mathbf{r}_{A/B}|/\Delta t) = \lim_{\Delta t \to 0} (r\Delta\theta/\Delta t)$ which, with $\omega = \dot{\theta}$, becomes

$$v_{A/B} = r\omega \qquad \textbf{(5/5)}$$

Using $\mathbf{r}$ to represent the vector $\mathbf{r}_{A/B}$ from the first of Eqs. 5/3, we may write the relative velocity as the vector

$$\mathbf{v}_{A/B} = \boldsymbol{\omega} \times \mathbf{r} \qquad \textbf{(5/6)}$$

where $\boldsymbol{\omega}$ is the angular velocity vector normal to the plane of the motion in the sense determined by the right-hand rule. A critical observation seen from Figs. 5/5*b* and *c* is that the relative linear velocity is always perpendicular to the line joining the two points in question.

Interpretation of the Relative-Velocity Equation

We can better understand the application of Eq. 5/4 by visualizing the separate translation and rotation components of the equation. These components are emphasized in Fig. 5/6, which shows a rigid body

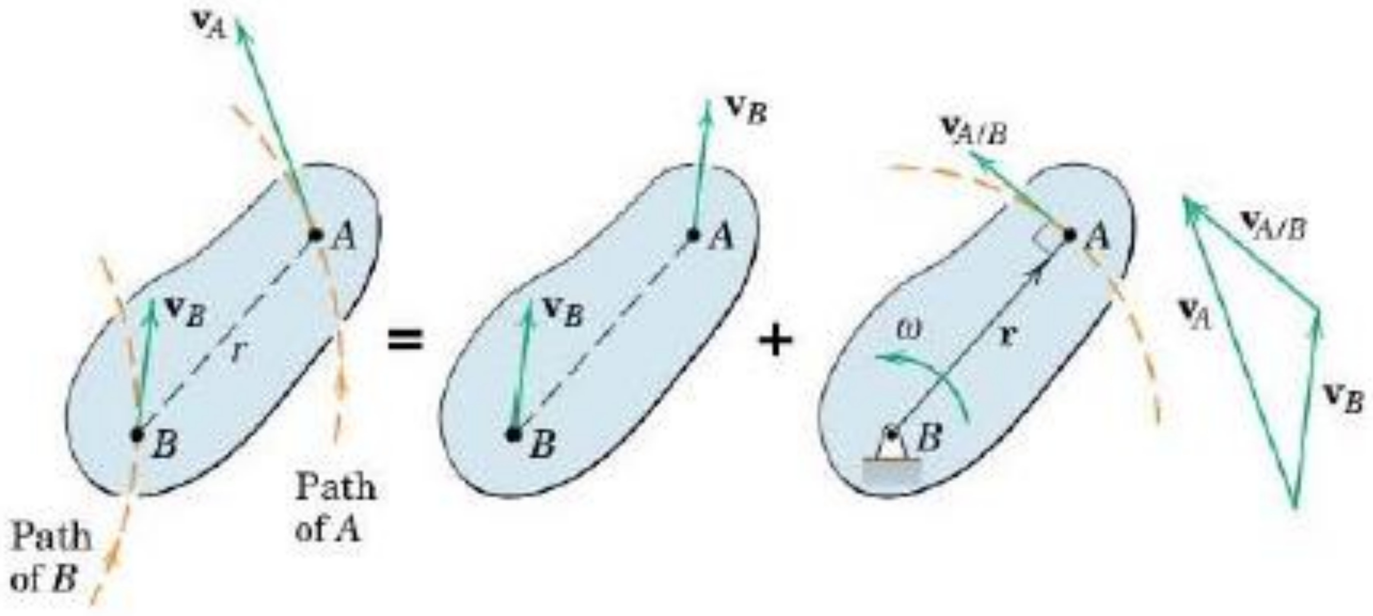


Figure 5/6

in plane motion. With B chosen as the reference point, the velocity of A is the vector sum of the translational portion $\mathbf{v}_B$, plus the rotational portion $\mathbf{v}_{A/B} = \boldsymbol{\omega} \times \mathbf{r}$, which has the magnitude $v_{A/B} = r\omega$, where $|\boldsymbol{\omega}| = \dot{\theta}$, the *absolute* angular velocity of AB. The fact that the *relative linear velocity* is *always perpendicular* to the line joining the two points in question is an important key to the solution of many problems. To reinforce your understanding of this concept, you should draw the equivalent diagram where point A is used as the reference point rather than B.

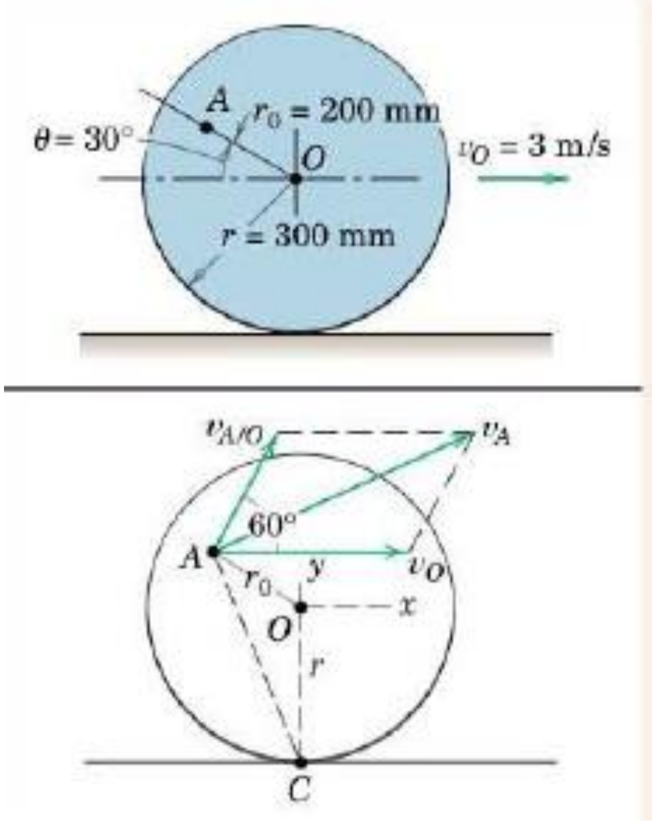
Equation 5/4 may also be used to analyze constrained sliding contact between two links in a mechanism. In this case, we choose points A and B as coincident points, one on each link, for the instant under consideration. In contrast to the previous example, in this case, the two points are on different bodies so they are not a fixed distance apart. This second use of the relative-velocity equation is illustrated in Sample Problem 5/10.

Solution of the Relative-Velocity Equation

Solution of the relative-velocity equation may be carried out by scalar or vector algebra, or a graphical analysis may be employed. A sketch of the vector polygon which represents the vector equation should always be made to reveal the physical relationships involved. From this sketch, you can write scalar component equations by projecting the vectors along convenient directions. You can usually avoid solving simultaneous equations by a careful choice of the projections. Alternatively, each term in the relative-motion equation may be written in terms of its **i**- and **j**-components, from which you will obtain two scalar equations when the equality is applied, separately, to the coefficients of the **i**- and **j**-terms.

Sample Problem 5/7

The wheel of radius $r = 300$ mm rolls to the right without slipping and has a velocity $v_O = 3$ m/s of its center O. Calculate the velocity of point A on the wheel for the instant represented.



Solution I (Scalar-Geometric). The center O is chosen as the reference point for the relative-velocity equation since its motion is given. We therefore write

$$\mathbf{v}_A = \mathbf{v}_O + \mathbf{v}_{A/O}$$

where the relative-velocity term is observed from the translating axes x-y attached to O. The angular velocity of AO is the same as that of the wheel which, from Sample Problem 5/4, is $\omega = v_O/r = 3/0.3 = 10$ rad/s. Thus, from Eq. 5/5 we have

$[v_{A/O} = r_0\dot{\theta}]$ $\qquad v_{A/O} = 0.2(10) = 2$ m/s

which is normal to AO as shown. The vector sum $\mathbf{v}_A$ is shown on the diagram and may be calculated from the law of cosines. Thus,

$$v_A{}^2 = 3^2 + 2^2 + 2(3)(2)\cos 60^\circ = 19\ (\text{m/s})^2 \qquad v_A = 4.36\ \text{m/s} \qquad \textit{Ans.}$$

The contact point C momentarily has zero velocity and can be used alternatively as the reference point, in which case, the relative-velocity equation becomes $\mathbf{v}_A = \mathbf{v}_C + \mathbf{v}_{A/C} = \mathbf{v}_{A/C}$ where

$$v_{A/C} = \overline{AC}\omega = \frac{\overline{AC}}{\overline{OC}}v_O = \frac{0.436}{0.300}(3) = 4.36\ \text{m/s} \qquad v_A = v_{A/C} = 4.36\ \text{m/s}$$

The distance $\overline{AC} = 436$ mm is calculated separately. We see that $\mathbf{v}_A$ is normal to AC since A is momentarily rotating about point C.

Solution II (Vector). We will now use Eq. 5/6 and write

$$\mathbf{v}_A = \mathbf{v}_O + \mathbf{v}_{A/O} = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}_0$$

where

$$\boldsymbol{\omega} = -10\mathbf{k} \text{ rad/s}$$

$$\mathbf{r}_0 = 0.2(-\mathbf{i}\cos 30° + \mathbf{j}\sin 30°) = -0.1732\mathbf{i} + 0.1\mathbf{j} \text{ m}$$

$$\mathbf{v}_O = 3\mathbf{i} \text{ m/s}$$

We now solve the vector equation

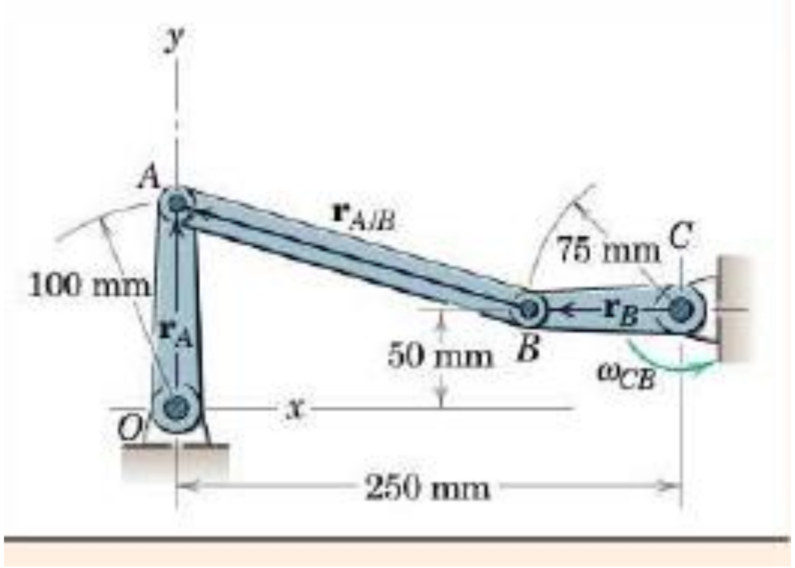
$$\mathbf{v}_A = 3\mathbf{i} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -10 \\ -0.1732 & 0.1 & 0 \end{vmatrix} = 3\mathbf{i} + 1.732\mathbf{j} + \mathbf{i}$$

$$= 4\mathbf{i} + 1.732\mathbf{j} \text{ m/s} \qquad \textit{Ans.}$$

The magnitude $v_A = \sqrt{4^2 + (1.732)^2} = \sqrt{19} = 4.36$ m/s and direction agree with the previous solution.

Sample Problem 5/8

Crank CB oscillates about C through a limited arc, causing crank OA to oscillate about O. When the linkage passes the position shown with CB horizontal and OA vertical, the angular velocity of CB is 2 rad/s counterclockwise. For this instant, determine the angular velocities of OA and AB.



Solution I (Vector). The relative-velocity equation $\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$ is rewritten as

$$\boldsymbol{\omega}_{OA} \times \mathbf{r}_A = \boldsymbol{\omega}_{CB} \times \mathbf{r}_B + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/B}$$

where

$$\boldsymbol{\omega}_{OA} = \omega_{OA}\mathbf{k} \qquad \boldsymbol{\omega}_{CB} = 2\mathbf{k} \text{ rad/s} \qquad \boldsymbol{\omega}_{AB} = \omega_{AB}\mathbf{k}$$

$$\mathbf{r}_A = 100\mathbf{j} \text{ mm} \qquad \mathbf{r}_B = -75\mathbf{i} \text{ mm} \qquad \mathbf{r}_{A/B} = -175\mathbf{i} + 50\mathbf{j} \text{ mm}$$

Substitution gives

$$\omega_{OA}\mathbf{k} \times 100\mathbf{j} = 2\mathbf{k} \times (-75\mathbf{i}) + \omega_{AB}\mathbf{k} \times (-175\mathbf{i} + 50\mathbf{j})$$

$$-100\omega_{OA}\mathbf{i} = -150\mathbf{j} - 175\omega_{AB}\mathbf{j} - 50\omega_{AB}\mathbf{i}$$

Matching coefficients of the respective **i**- and **j**-terms gives

$$-100\omega_{OA} + 50\omega_{AB} = 0 \qquad 25(6 + 7\omega_{AB}) = 0$$

the solutions of which are

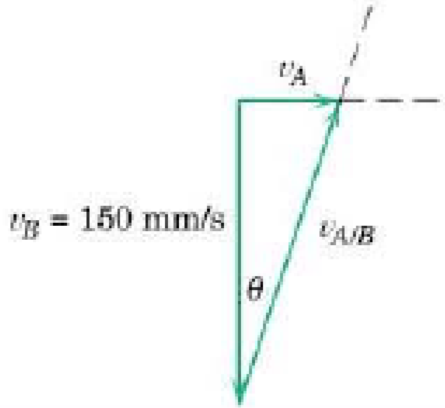
$$\omega_{AB} = -6/7 \text{ rad/s} \qquad \text{and} \qquad \omega_{OA} = -3/7 \text{ rad/s} \qquad \textit{Ans.}$$

Solution II (Scalar-Geometric). Solution by the scalar geometry of the vector triangle is particularly simple here since $\mathbf{v}_A$ and $\mathbf{v}_B$ are at right angles for this special position of the linkages. First, we compute v_B, which is

$[v = r\omega]$ $$v_B = 0.075(2) = 0.150 \text{ m/s}$$

and represent it in its correct direction as shown. The vector $\mathbf{v}_{A/B}$ must be perpendicular to AB, and the angle θ between $\mathbf{v}_{A/B}$ and $\mathbf{v}_B$ is also the angle made by AB with the horizontal direction. This angle is given by

$$\tan\theta = \frac{100 - 50}{250 - 75} = \frac{2}{7}$$



The horizontal vector $\mathbf{v}_A$ completes the triangle for which we have

$$v_{A/B} = v_B/\cos\theta = 0.150/\cos\theta$$

$$v_A = v_B \tan\theta = 0.150(2/7) = 0.30/7 \text{ m/s}$$

The angular velocities become

$[\omega = v/r]$ $$\omega_{AB} = \frac{v_{A/B}}{AB} = \frac{0.150}{\cos\theta}\,\frac{\cos\theta}{0.250 - 0.075}$$

$$= 6/7 \text{ rad/s CW}$$ *Ans.*

$[\omega = v/r]$ $$\omega_{OA} = \frac{v_A}{OA} = \frac{0.30}{7}\,\frac{1}{0.100} = 3/7 \text{ rad/s CW}$$ *Ans.*

5/5 Instantaneous Center of Zero Velocity

In the previous article, we determined the velocity of a point on a rigid body in plane motion by adding the relative velocity due to rotation about a convenient reference point to the velocity of the reference point. We now solve the problem by choosing a unique reference point which momentarily has zero velocity. As far as velocities are concerned, the body may be considered to be in pure rotation about an axis, normal to the plane of motion, passing through this point. This axis is called the *instantaneous axis of zero velocity*, and the intersection of this axis with the plane of motion is known as the *instantaneous center of zero velocity*. This approach provides us with a valuable means for visualizing and analyzing velocities in plane motion.

Locating the Instantaneous Center

The existence of the *instantaneous center* is easily shown. For the body in Fig. 5/7, assume that the directions of the absolute velocities of any two points A and B on the body are known and are not parallel. If there is a point about which A has absolute circular motion at the instant considered, this point must lie on the normal to $\mathbf{v}_A$ through A. Similar reasoning applies to B, and the intersection of the two perpendiculars fulfills the requirement for an absolute center of rotation *at the instant considered*. Point C is the instantaneous center of zero velocity and may lie on or off the body. If it lies off the body, it may be visualized as lying on an imaginary extension of the body. The instantaneous center need not be a fixed point in the body or a fixed point in the plane.

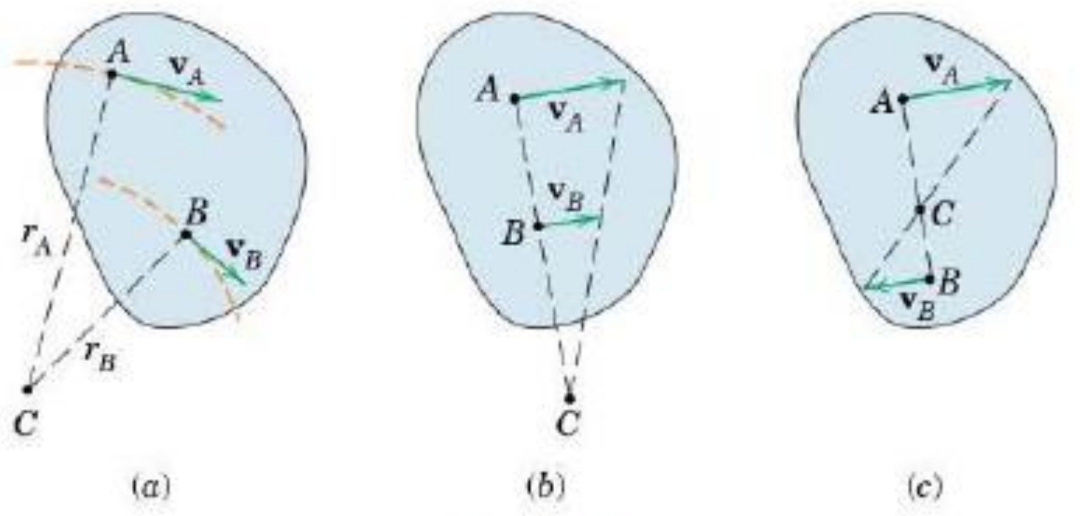

A
$\mathbf{v}_A$
B
$\mathbf{v}_B$
r_A
r_B
C
(a)
A
$\mathbf{v}_A$
$\mathbf{v}_B$
B
C
(b)
$\mathbf{v}_A$
A
C
B
$\mathbf{v}_B$
(c)

Figure 5/7

If we also know the magnitude of the velocity of one of the points, say, v_A, we may easily obtain the angular velocity ω of the body and the linear velocity of every point in the body. Thus, the angular velocity of the body, Fig. 5/7*a*, is

$$\omega = \frac{v_A}{r_A}$$

which, of course, is also the angular velocity of *every* line in the body. Therefore, the velocity of *B* is $v_B = r_B\omega = (r_B/r_A)v_A$. Once the instantaneous center is located, the direction of the instantaneous velocity of

every point in the body is readily found since it must be perpendicular to the radial line joining the point in question with C.

Motion of the Instantaneous Center

As the body changes its position, the instantaneous center C also changes its position both in space and on the body. The locus of the instantaneous centers in space is known as the *space centrode*, and the locus of the positions of the instantaneous centers on the body is known as the *body centrode*. At the instant considered, the two curves are tangent at the position of point C. It can be shown that the body-centrode curve rolls on the space-centrode curve during the motion of the body, as indicated schematically in Fig. 5/8.

Although the instantaneous center of zero velocity is momentarily at rest, its acceleration generally is *not* zero. Thus, this point may *not* be used as an instantaneous center of zero acceleration in a manner analogous to its use for finding velocity. An instantaneous center of zero acceleration does exist for bodies in general plane motion, but its location and use represent a specialized topic in mechanism kinematics and will not be treated here.

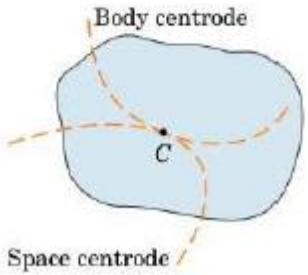
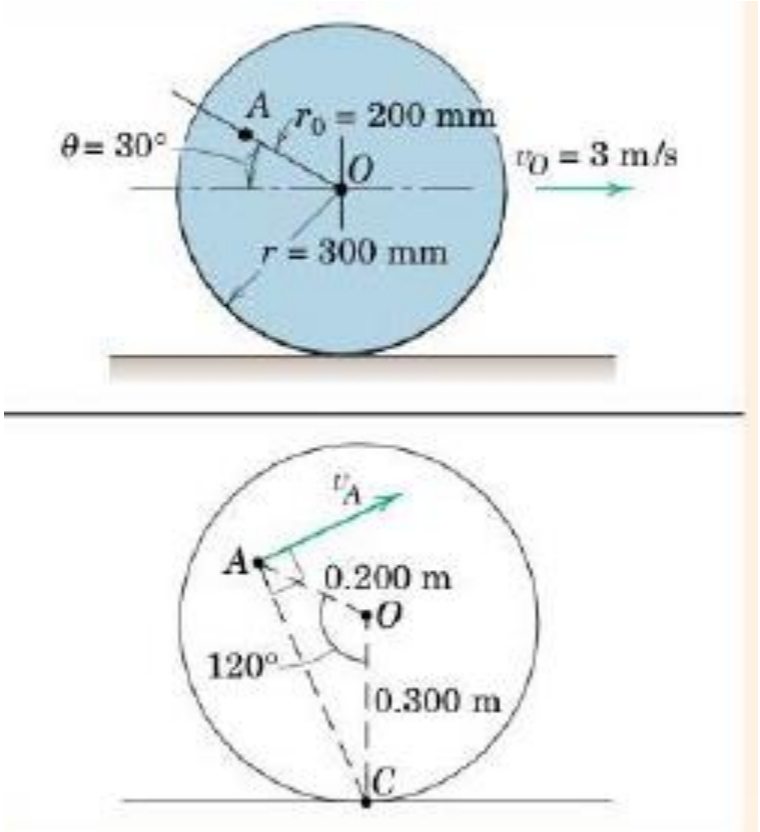


Figure 5/8

Sample Problem 5/11

The wheel of Sample Problem 5/7, shown again here, rolls to the right without slipping, with its center O having a velocity $v_O = 3$ m/s. Locate the instantaneous center of zero velocity and use it to find the velocity of point A for the position indicated.



Solution. The point on the rim of the wheel in contact with the ground has no velocity if the wheel is not slipping; it is, therefore, the instantaneous center C of zero velocity. The angular velocity of the wheel becomes

$[\omega = v/r]$ $$\omega = v_O/\overline{OC} = 3/0.300 = 10 \text{ rad/s}$$

The distance from A to C is

$$\overline{AC} = \sqrt{(0.300)^2 + (0.200)^2 - 2(0.300)(0.200)\cos 120^\circ} = 0.436 \text{ m}$$

The velocity of A becomes

$[v = r\omega]$ $$v_A = \overline{AC}\omega = 0.436(10) = 4.36 \text{ m/s}$$ *Ans.*

The direction of $\mathbf{v}_A$ is perpendicular to AC as shown.

5/6 RELATIVE ACCELERATION

Consider the equation $\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$, which describes the relative velocities of two points A and B in plane motion in terms of nonrotating reference axes. By differentiating the equation with respect to time, we may obtain the relative-acceleration equation, which is $\dot{\mathbf{v}}_A = \dot{\mathbf{v}}_B + \dot{\mathbf{v}}_{A/B}$ or

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B} \qquad (5/7)$$

In words, Eq. 5/7 states that the acceleration of point A equals the vector sum of the acceleration of point B and the acceleration which A appears to have to a nonrotating observer moving with B.

Relative Acceleration Due to Rotation

If points A and B are located on the same rigid body and in the plane of motion, the distance r between them remains constant so that the observer moving with B perceives A to have circular motion about B, as we saw in Art. 5/4 with the relative-velocity relationship. Because the relative motion is circular, it follows that the relative-acceleration term will have both a normal component directed from A toward B due to the change of direction of $\mathbf{v}_{A/B}$ and a tangential component perpendicular to AB due to the change in magnitude of $\mathbf{v}_{A/B}$. These acceleration components for circular motion, cited in Eqs. 5/2, were covered earlier in Art. 2/5 and should be thoroughly familiar by now.

Thus we may write

$$\mathbf{a}_A = \mathbf{a}_B + (\mathbf{a}_{A/B})_n + (\mathbf{a}_{A/B})_t \qquad \textbf{(5/8)}$$

where the magnitudes of the relative-acceleration components are

$$(a_{A/B})_n = v_{A/B}{}^2/r = r\omega^2$$
$$(a_{A/B})_t = \dot{v}_{A/B} = r\alpha \qquad \textbf{(5/9)}$$

In vector notation the acceleration components are

$$(\mathbf{a}_{A/B})_n = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$
$$(\mathbf{a}_{A/B})_t = \boldsymbol{\alpha} \times \mathbf{r} \tag{5/9a}$$

In these relationships, $\boldsymbol{\omega}$ is the angular velocity and $\boldsymbol{\alpha}$ is the angular acceleration of the body. The vector locating A from B is $\mathbf{r}$. It is important to observe that the *relative* acceleration terms depend on the respective *absolute* angular velocity and *absolute* angular acceleration.

Alternatively, we may express the acceleration of B in terms of the acceleration of A, which puts the nonrotating reference axes on A rather than B. This order gives

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

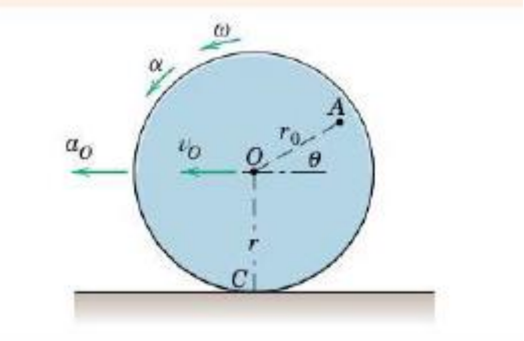
Here $\mathbf{a}_{B/A}$ and its n- and t-components are the negatives of $\mathbf{a}_{A/B}$ and its n- and t-components. To understand this analysis better, you should make a sketch corresponding to Fig. 5/9 for this choice of terms.

Solution of the Relative-Acceleration Equation

As in the case of the relative-velocity equation, we can handle the solution to Eq. 5/8 in three different ways, namely, by scalar algebra and geometry, by vector algebra, or by graphical construction. It is helpful to be familiar with all three techniques. You should make a sketch of the vector polygon representing the vector equation and pay close attention to the head-to-tail combination of vectors so that it agrees with the equation. Known vectors should be added first, and the unknown vectors will become the closing legs of the vector polygon. It is vital that you visualize the vectors in their geometrical sense, as only then can you understand the full significance of the acceleration equation.

Sample Problem 5/13

The wheel of radius r rolls to the left without slipping and, at the instant considered, the center O has a velocity $\mathbf{v}_O$ and an acceleration $\mathbf{a}_O$ to the left. Determine the acceleration of points A and C on the wheel for the instant considered.



Solution. From our previous analysis of Sample Problem 5/4, we know that the angular velocity and angular acceleration of the wheel are

$$\omega = v_O/r \qquad \text{and} \qquad \alpha = a_O/r$$

The acceleration of A is written in terms of the given acceleration of O. Thus,

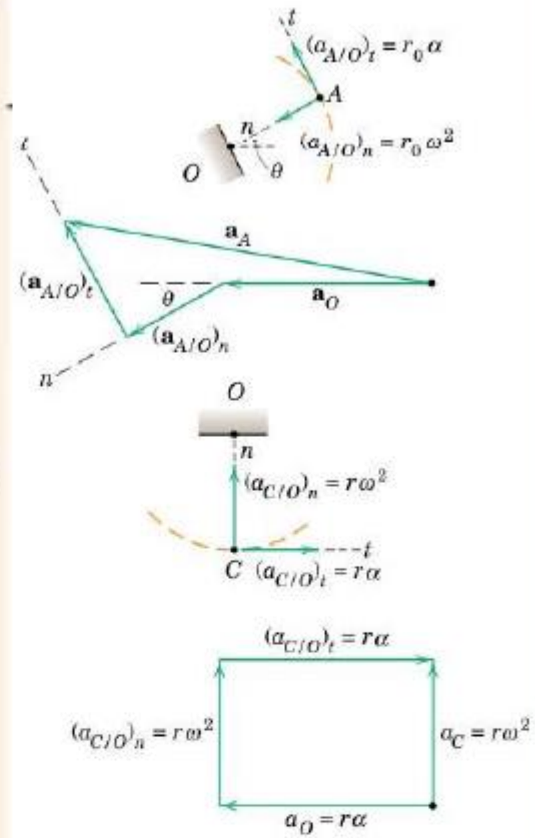
$$\mathbf{a}_A = \mathbf{a}_O + \mathbf{a}_{A/O} = \mathbf{a}_O + (\mathbf{a}_{A/O})_n + (\mathbf{a}_{A/O})_t$$

The relative-acceleration terms are viewed as though O were fixed, and for this relative circular motion they have the magnitudes

$$(a_{A/O})_n = r_0\omega^2 = r_0\left(\frac{v_O}{r}\right)^2$$

$$(a_{A/O})_t = r_0\alpha = r_0\left(\frac{a_O}{r}\right)$$

① and the directions shown.

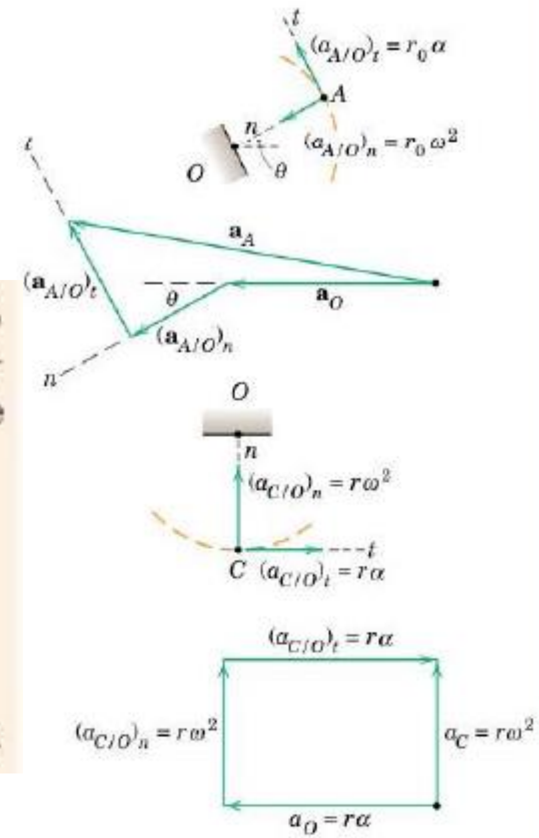


Adding the vectors head-to-tail gives $\mathbf{a}_A$ as shown. In a numerical problem, we may obtain the combination algebraically or graphically. The algebraic expression for the magnitude of $\mathbf{a}_A$ is found from the square root of the sum of the squares of its components. If we use n- and t-directions, we have

② $$a_A = \sqrt{(a_A)_n{}^2 + (a_A)_t{}^2}$$

$$= \sqrt{[a_O \cos\theta + (a_{A/O})_n]^2 + [a_O \sin\theta + (a_{A/O})_t]^2}$$

$$= \sqrt{(r\alpha \cos\theta + r_0\omega^2)^2 + (r\alpha \sin\theta + r_0\alpha)^2} \qquad \textit{Ans.}$$



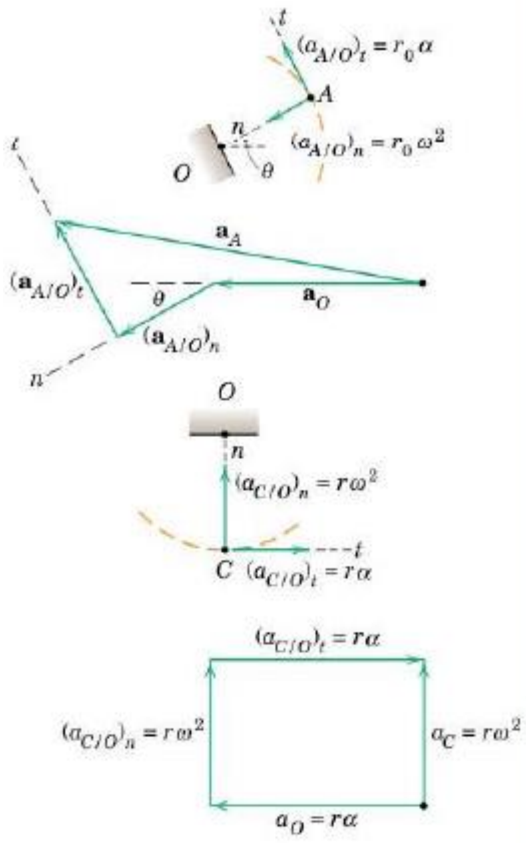
The direction of $\mathbf{a}_A$ can be computed if desired.

The acceleration of the instantaneous center C of zero velocity, considered a point on the wheel, is obtained from the expression

$$\mathbf{a}_C = \mathbf{a}_O + \mathbf{a}_{C/O}$$

where the components of the relative-acceleration term are $(a_{C/O})_n = r\omega^2$ directed from C to O and $(a_{C/O})_t = r\alpha$ directed to the right because of the counterclockwise angular acceleration of line CO about O. The terms are added together in the lower diagram and it is seen that

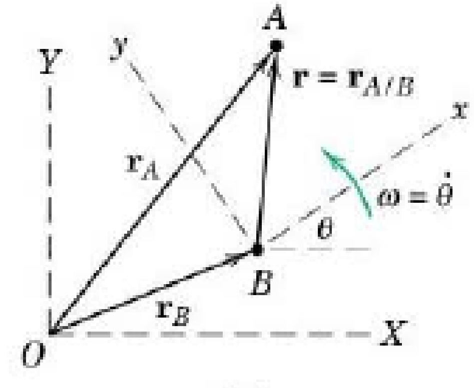
③ $$a_C = r\omega^2$$ *Ans.*



5/7 MOTION RELATIVE TO ROTATING AXES

In our discussion of the relative motion of particles in Art. 2/8 and in our use of the relative-motion equations for the plane motion of rigid bodies in this present chapter, we have used *nonrotating* reference axes to describe relative velocity and relative acceleration. Use of rotating reference axes greatly facilitates the solution of many problems in kinematics where motion is generated within a system or observed from a system which itself is rotating. An example of such a motion is the movement of a fluid particle along the curved vane of a centrifugal pump, where the path relative to the vanes of the impeller becomes an important design consideration.

We begin the description of motion using rotating axes by considering the plane motion of two particles A and B in the fixed X-Y plane, Fig. 5/10a. For the time being, we will consider A and B to be moving independently of one another for the sake of generality. We observe the motion of A from a moving reference frame x-y which has its origin attached to B and which rotates with an angular velocity $\omega = \dot{\theta}$. We may write this angular velocity as the vector $\boldsymbol{\omega} = \omega\mathbf{k} = \dot{\theta}\mathbf{k}$, where the vector is normal to the plane of motion and where its positive sense is in the positive z-direction (out from the paper), as established by the right-hand rule. The absolute position vector of A is given by



(a)

$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r} = \mathbf{r}_B + (x\mathbf{i} + y\mathbf{j}) \qquad \textbf{(5/10)}$$

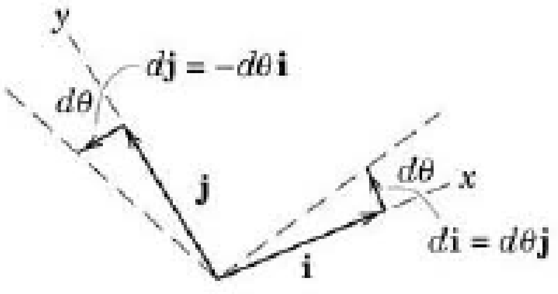
where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors attached to the x-y frame and $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ stands for $\mathbf{r}_{A/B}$, the position vector of A with respect to B.

Time Derivatives of Unit Vectors

To obtain the velocity and acceleration equations we must successively differentiate the position-vector equation with respect to time. In contrast to the case of translating axes treated in Art. 2/8, the unit vectors **i** and **j** are now rotating with the x-y axes and, therefore, have time derivatives which must be evaluated. These derivatives may be seen from Fig. 5/10b, which shows the infinitesimal change in each unit vector during time dt as the reference axes rotate through an angle $d\theta = \omega\, dt$. The differential change in **i** is $d\mathbf{i}$, and it has the direction of **j** and a magnitude equal to the angle $d\theta$ times the length of the vector **i**, which is unity. Thus, $d\mathbf{i} = d\theta\, \mathbf{j}$.

Similarly, the unit vector **j** has an infinitesimal change $d\mathbf{j}$ which points in the negative x-direction, so that $d\mathbf{j} = -d\theta\, \mathbf{i}$. Dividing by dt and replacing $d\mathbf{i}/dt$ by $\dot{\mathbf{i}}$, $d\mathbf{j}/dt$ by $\dot{\mathbf{j}}$, and $d\theta/dt$ by $\dot{\theta} = \omega$ result in

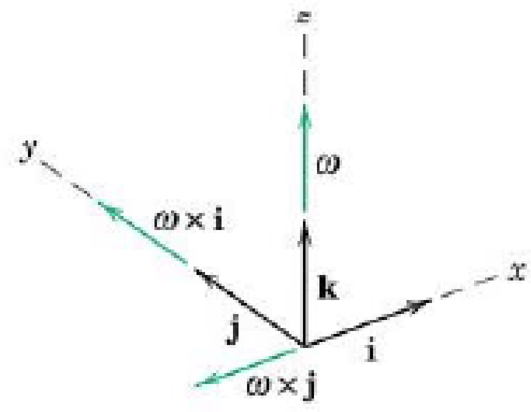
$$\dot{\mathbf{i}} = \omega \mathbf{j} \qquad \text{and} \qquad \dot{\mathbf{j}} = -\omega \mathbf{i}$$



(b)

By using the cross product, we can see from Fig. 5/10c that $\boldsymbol{\omega} \times \mathbf{i} = \omega\mathbf{j}$ and $\boldsymbol{\omega} \times \mathbf{j} = -\omega\mathbf{i}$. Thus, the time derivatives of the unit vectors may be written as

$$\dot{\mathbf{i}} = \boldsymbol{\omega} \times \mathbf{i} \qquad \text{and} \qquad \dot{\mathbf{j}} = \boldsymbol{\omega} \times \mathbf{j} \tag{5/11}$$



(c)

Figure 5/10

Relative Velocity

We now use the expressions of Eqs. 5/11 when taking the time derivative of the position-vector equation for A and B to obtain the relative-velocity relation. Differentiation of Eq. 5/10 gives

$$\dot{\mathbf{r}}_A = \dot{\mathbf{r}}_B + \frac{d}{dt}(x\mathbf{i} + y\mathbf{j})$$

$$= \dot{\mathbf{r}}_B + (x\dot{\mathbf{i}} + y\dot{\mathbf{j}}) + (\dot{x}\mathbf{i} + \dot{y}\mathbf{j})$$

But $x\dot{\mathbf{i}} + y\dot{\mathbf{j}} = \boldsymbol{\omega} \times x\mathbf{i} + \boldsymbol{\omega} \times y\mathbf{j} = \boldsymbol{\omega} \times (x\mathbf{i} + y\mathbf{j}) = \boldsymbol{\omega} \times \mathbf{r}$. Also, since the observer in x-y measures velocity components $\dot{x}$ and $\dot{y}$, we see that $\dot{x}\mathbf{i} + \dot{y}\mathbf{j} = \mathbf{v}_{\text{rel}}$, which is the velocity relative to the x-y frame of reference. Thus, the relative-velocity equation becomes

$$\boxed{\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{\text{rel}}} \qquad \textbf{(5/12)}$$

Comparison of Eq. 5/12 with Eq. 2/20 for nonrotating reference axes shows that $\mathbf{v}_{A/B} = \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{rel}$, from which we conclude that the term $\boldsymbol{\omega} \times \mathbf{r}$ is the difference between the relative velocities as measured from nonrotating and rotating axes.

The following comparison will help establish the equivalence of, and clarify the differences between, the relative-velocity equations written for rotating and nonrotating reference axes:

$$
\begin{aligned}
\mathbf{v}_A &= \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{\text{rel}} \\
\mathbf{v}_A &= \underbrace{\mathbf{v}_B + \mathbf{v}_{P/B}} + \mathbf{v}_{A/P} \\
\mathbf{v}_A &= \qquad \mathbf{v}_P \qquad \underbrace{\qquad + \mathbf{v}_{A/P}} \\
\mathbf{v}_A &= \mathbf{v}_B + \qquad \mathbf{v}_{A/B}
\end{aligned}
\qquad \mathbf{(5/12}\boldsymbol{a}\mathbf{)}
$$

References

- J.L. Meriam and L. G. Krage, Dynamics 6TH edition