MEC3705 – DYNAMICS

PART II

DYNAMICS OF RIGID BODIES

KINETICS OF RIGID BODIES

6

Plane Kinetics of Rigid Bodies

CHAPTER OUTLINE

6/1 INTRODUCTION

The *kinetics* of rigid bodies treats the relationships between the external forces acting on a body and the corresponding translational and rotational motions of the body. In Chapter 5 we developed the kinematic relationships for the plane motion of rigid bodies, and we will use these relationships extensively in this present chapter, where the effects of forces on the two-dimensional motion of rigid bodies are examined.

Background for the Study of Kinetics

In Chapter 3 we found that two force equations of motion were required to define the motion of a particle whose motion is confined to a plane. For the plane motion of a rigid body, an additional equation is needed to specify the state of rotation of the body. Thus, two force equations and one moment equation or their equivalent are required to determine the state of rigid-body plane motion.

The kinetic relationships which form the basis for most of the analysis of rigid-body motion were developed in Chapter 4 for a general system of particles. Frequent reference will be made to these equations as they are further developed in Chapter 6 and applied specifically to the plane motion of rigid bodies. You should refer to Chapter 4 frequently as you study Chapter 6. Also, before proceeding make sure that you have a firm grasp of the calculation of velocities and accelerations as developed in Chapter 5 for rigid-body plane motion. Unless you can determine accelerations correctly from the principles of kinematics, you frequently will be unable to apply the force and moment principles of kinetics. Consequently, you should master the necessary kinematics, including the calculation of relative accelerations, before proceeding.

In the kinetics of rigid bodies which have angular motion, we must introduce a property of the body which accounts for the radial distribution of its mass with respect to a particular axis of rotation normal to the plane of motion. This property is known as the *mass moment of inertia* of the body, and it is essential that we be able to calculate this property in order to solve rotational problems. We assume that you are familiar with the calculation of mass moments of inertia. Appendix B treats this topic for those who need instruction or review.

6/2 General Equations of Motion

In Arts. 4/2 and 4/4 we derived the force and moment vector equations of motion for a general system of mass. We now apply these results by starting, first, with a general rigid body in three dimensions. The force equation, Eq. 4/1,

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}} \qquad [4/1]$$

tells us that the resultant $\Sigma \mathbf{F}$ of the external forces acting on the body equals the mass m of the body times the acceleration $\bar{\mathbf{a}}$ of its mass center G. The moment equation taken about the mass center, Eq. 4/9,

$$\Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G \qquad [4/9]$$

shows that the resultant moment about the mass center of the external forces on the body equals the time rate of change of the angular momentum of the body about the mass center.

Recall from our study of statics that a general system of forces acting on a rigid body may be replaced by a resultant force applied at a chosen point and a corresponding couple. By replacing the external forces by their equivalent force-couple system in which the resultant force acts through the mass center, we may visualize the action of the forces and the corresponding dynamic response of the body with the aid of Fig. 6/1.

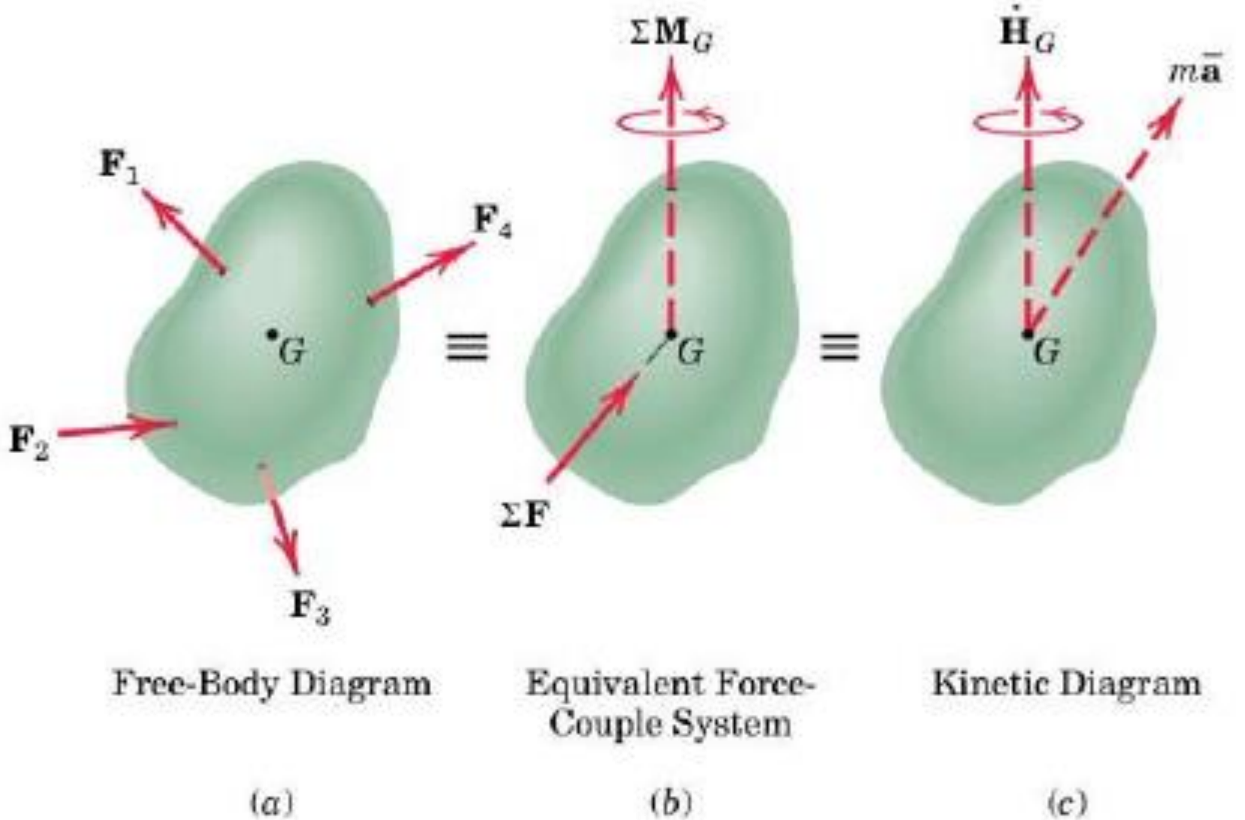


Figure 6/1

Part *a* of the figure shows the relevant free-body diagram. Part *b* of the figure shows the equivalent force-couple system with the resultant force applied through G. Part *c* of the figure is a *kinetic diagram*, which represents the resulting dynamic effects as specified by Eqs. 4/1 and 4/9. The equivalence between the free-body diagram and the kinetic diagram enables us to clearly visualize and easily remember the separate translational and rotational effects of the forces applied to a rigid body. We will express this equivalence mathematically as we apply these results to the treatment of rigid-body plane motion.

Plane-Motion Equations

We now apply the foregoing relationships to the case of plane motion. Figure 6/2 represents a rigid body moving with plane motion in the x-y plane. The mass center G has an acceleration $\bar{\mathbf{a}}$, and the body has an angular velocity $\boldsymbol{\omega} = \omega\mathbf{k}$ and an angular acceleration $\boldsymbol{\alpha} = \alpha\mathbf{k}$, both taken positive in the z-direction. Because the z-direction of both $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ remains perpendicular to the plane of motion, we may use scalar notation ω and $\alpha = \dot{\omega}$ to represent the angular velocity and angular acceleration.

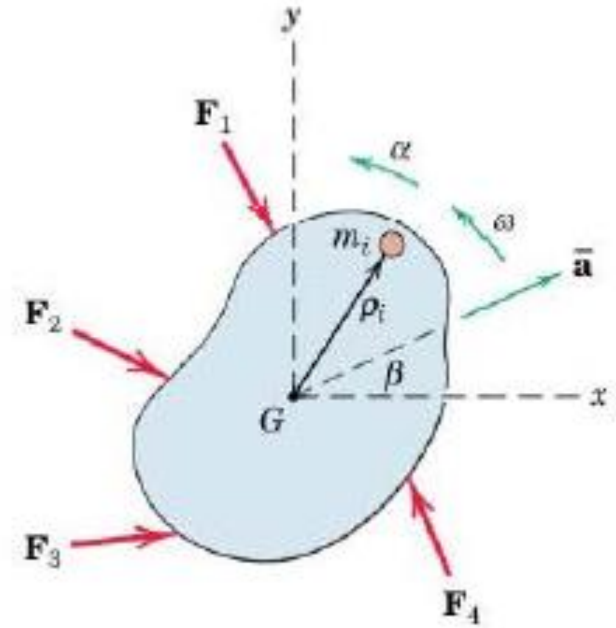


Figure 6/2

The angular momentum about the mass center for the general system was expressed in Eq. 4/8*a* as $\mathbf{H}_G = \Sigma \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i$ where $\boldsymbol{\rho}_i$ is the position vector relative to G of the representative particle of mass m_i. For our rigid body, the velocity of m_i relative to G is $\dot{\boldsymbol{\rho}}_i = \boldsymbol{\omega} \times \boldsymbol{\rho}_i$, which has a magnitude $\rho_i\omega$ and lies in the plane of motion normal to $\boldsymbol{\rho}_i$. The product $\boldsymbol{\rho}_i \times \dot{\boldsymbol{\rho}}_i$ is then a vector normal to the x-y plane in the sense of $\boldsymbol{\omega}$, and its magnitude is $\rho_i^2\omega$. Thus, the magnitude of $\mathbf{H}_G$ becomes $H_G = \Sigma \rho_i^2 m_i \omega = \omega \Sigma \rho_i^2 m_i$. The summation, which may also be written as $\int \rho^2 \, dm$, is defined as the *mass moment of inertia* $\bar{I}$ of the body about the z-axis through G. (See Appendix B for a discussion of the calculation of mass moments of inertia.)

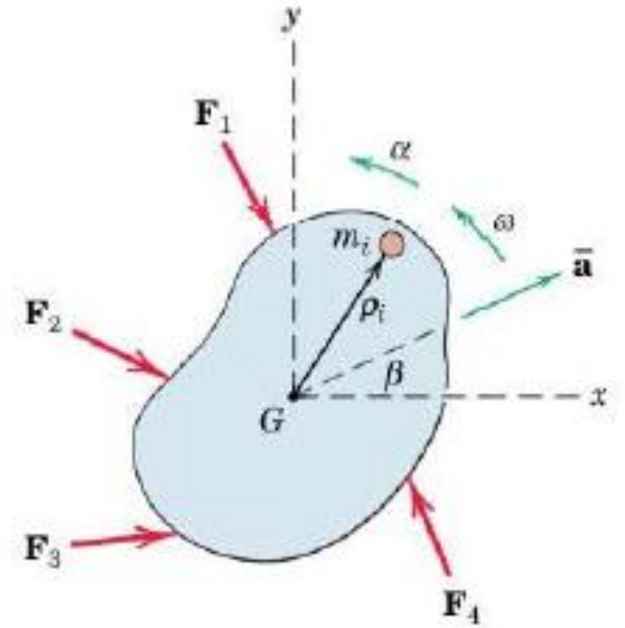


Figure 6/2

We may now write

$$H_G = \bar{I}\omega$$

where $\bar{I}$ is a constant property of the body. This property is a measure of the rotational inertia, which is the resistance to change in rotational velocity due to the radial distribution of mass around the z-axis through G. With this substitution, our moment equation, Eq. 4/9, becomes

$$\Sigma M_G = \dot{H}_G = \bar{I}\dot{\omega} = \bar{I}\alpha$$

where $\alpha = \dot{\omega}$ is the angular acceleration of the body.

We may now express the moment equation and the vector form of the generalized Newton's second law of motion, Eq. 4/1, as

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}}$$
$$\Sigma M_G = \bar{I}\alpha \qquad \textbf{(6/1)}$$

Equations 6/1 are the general equations of motion for a rigid body in plane motion. In applying Eqs. 6/1, we express the vector force equation

in terms of its two scalar components using x-y, n-t, or r-θ coordinates, whichever is most convenient for the problem at hand.

Alternative Derivation

It is instructive to use an alternative approach to derive the moment equation by referring directly to the forces which act on the representative particle of mass m_i, as shown in Fig. 6/3. The acceleration of m_i equals the vector sum of $\bar{a}$ and the relative terms $\rho_i\omega^2$ and $\rho_i\alpha$, where the mass center G is used as the reference point. It follows that the resultant of all forces on m_i has the components $m_i\bar{a}$, $m_i\rho_i\omega^2$, and $m_i\rho_i\alpha$ in the directions shown. The sum of the moments of these force components about G in the sense of α becomes

$$M_{G_i} = m_i\rho_i^2\alpha + (m_i\bar{a}\sin\beta)x_i - (m_i\bar{a}\cos\beta)y_i$$

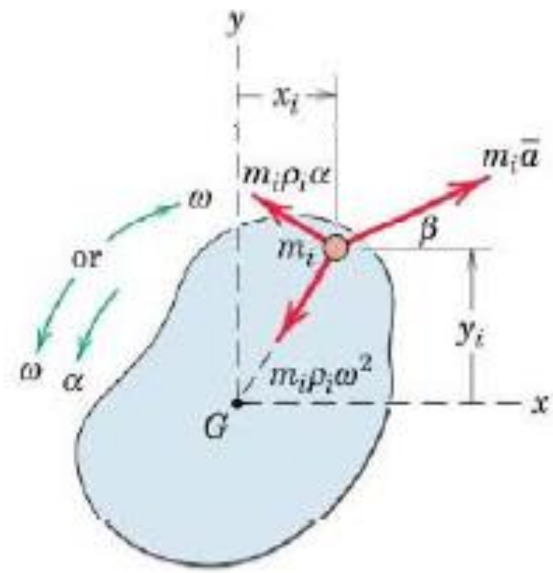


Figure 6/3

Similar moment expressions exist for all particles in the body, and the sum of these moments about G for the resultant forces acting on all particles may be written as

$$\Sigma M_G = \Sigma m_i \rho_i^2 \alpha + \bar{a} \sin\beta \, \Sigma m_i x_i - \bar{a} \cos\beta \, \Sigma m_i y_i$$

But the origin of coordinates is taken at the mass center, so that $\Sigma m_i x_i = m\bar{x} = 0$ and $\Sigma m_i y_i = m\bar{y} = 0$. Thus, the moment sum becomes

$$\Sigma M_G = \Sigma m_i \rho_i^2 \alpha = \bar{I}\alpha$$

as before. The contribution to ΣM_G of the forces internal to the body is, of course, zero since they occur in pairs of equal and opposite forces of action and reaction between interacting particles. Thus, ΣM_G, as before, represents the sum of moments about the mass center G of only the external forces acting on the body, as disclosed by the free-body diagram.

We note that the force component $m_i\rho_i\omega^2$ has no moment about G and conclude, therefore, that the angular velocity ω has no influence on the moment equation about the mass center.

The results embodied in our basic equations of motion for a rigid body in plane motion, Eqs. 6/1, are represented diagrammatically in Fig. 6/4,

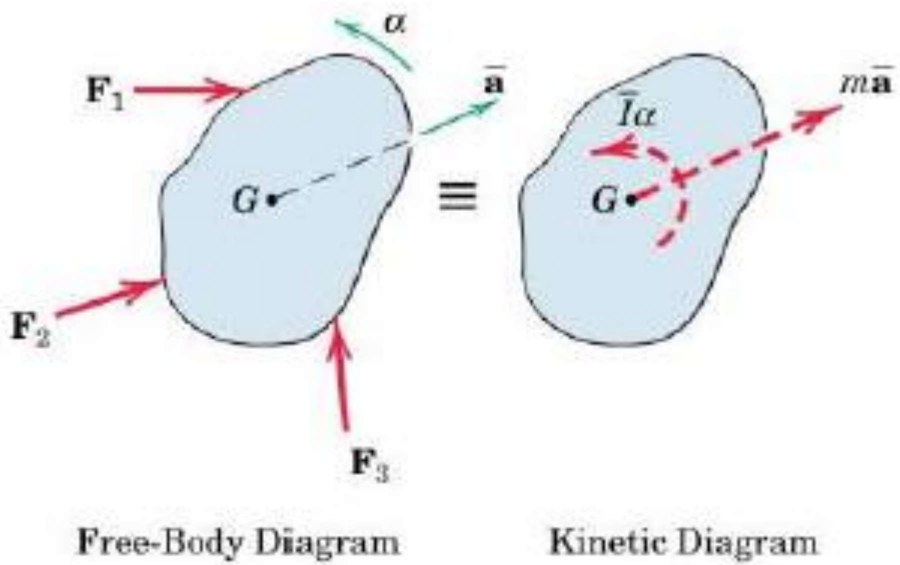


Figure 6/4

As previously mentioned, the translational term $m\bar{\mathbf{a}}$ will be expressed by its x-y, n-t, or r-θ components once the appropriate inertial reference system is designated. The equivalence depicted in Fig. 6/4 is basic to our understanding of the kinetics of plane motion and will be employed frequently in the solution of problems.

Representation of the resultants $m\bar{\mathbf{a}}$ and $\bar{I}\alpha$ will help ensure that the force and moment sums determined from the free-body diagram are equated to their proper resultants.

Alternative Moment Equations

In Art. 4/4 of Chapter 4 on systems of particles, we developed a general equation for moments about an arbitrary point P, Eq. 4/11, which is

$$\Sigma \mathbf{M}_P = \dot{\mathbf{H}}_G + \bar{\boldsymbol{\rho}} \times m\bar{\mathbf{a}} \qquad [4/11]$$

where $\bar{\boldsymbol{\rho}}$ is the vector from P to the mass center G and $\bar{\mathbf{a}}$ is the mass-center acceleration. As we have shown earlier in this article, for a rigid body in plane motion $\dot{\mathbf{H}}_G$ becomes $\bar{I}\alpha$. Also, the cross product $\bar{\boldsymbol{\rho}} \times m\bar{\mathbf{a}}$ is simply the moment of magnitude $m\bar{a}d$ of $m\bar{\mathbf{a}}$ about P. Therefore, for the two-dimensional body illustrated in Fig. 6/5 with its free-body diagram and kinetic diagram, we may rewrite Eq. 4/11 simply as

$$\Sigma M_P = \bar{I}\alpha + m\bar{a}d \qquad \mathbf{(6/2)}$$

Clearly, all three terms are positive in the counterclockwise sense for the example shown, and the choice of P eliminates reference to $\mathbf{F}_1$ and $\mathbf{F}_3$.

If we had wished to eliminate reference to $\mathbf{F}_2$ and $\mathbf{F}_3$, for example, by choosing their intersection as the reference point, then P would lie on the opposite side of the $m\bar{\mathbf{a}}$ vector, and the clockwise moment of $m\bar{\mathbf{a}}$

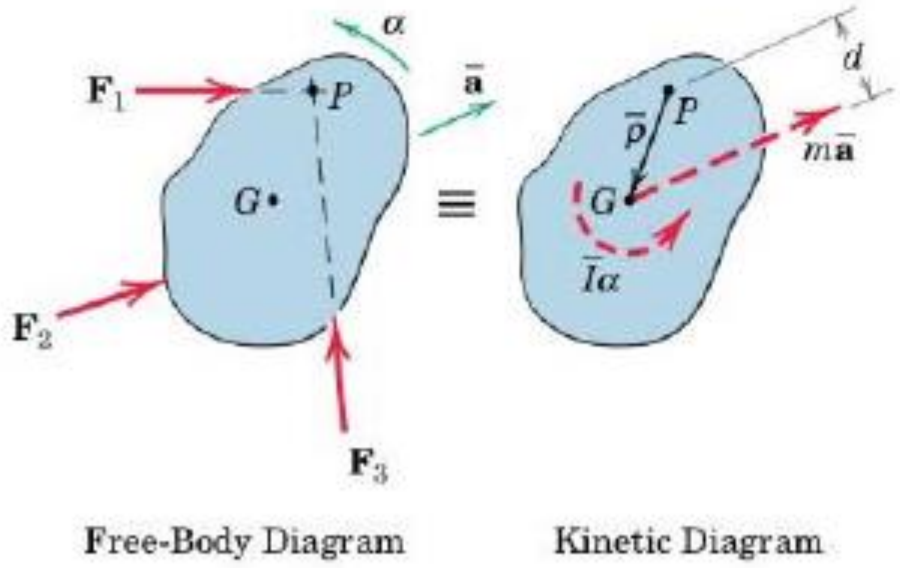


Figure 6/5

about P would be a negative term in the equation. Equation 6/2 is easily remembered as it is merely an expression of the familiar principle of moments, where the sum of the moments about P equals the combined moment about P of their sum, expressed by the resultant couple $\Sigma M_G = \bar{I}\alpha$ and the resultant force $\Sigma \mathbf{F} = m\bar{\mathbf{a}}$.

In Art. 4/4 we also developed an alternative moment equation about P, Eq. 4/13, which is

$$\Sigma \mathbf{M}_P = (\dot{\mathbf{H}}_P)_{rel} + \bar{\boldsymbol{\rho}} \times m\mathbf{a}_P \qquad \textbf{[4/13]}$$

For rigid-body plane motion, if P is chosen as a point *fixed* to the body, then in scalar form $(\dot{\mathbf{H}}_P)_{rel}$ becomes $I_P\alpha$, where I_P is the mass moment of inertia about an axis through P and α is the angular acceleration of the body. So we may write the equation as

$$\Sigma \mathbf{M}_P = I_P \boldsymbol{\alpha} + \bar{\boldsymbol{\rho}} \times m\mathbf{a}_P \qquad \textbf{(6/3)}$$

where the acceleration of P is $\mathbf{a}_P$ and the position vector from P to G is $\bar{\boldsymbol{\rho}}$.

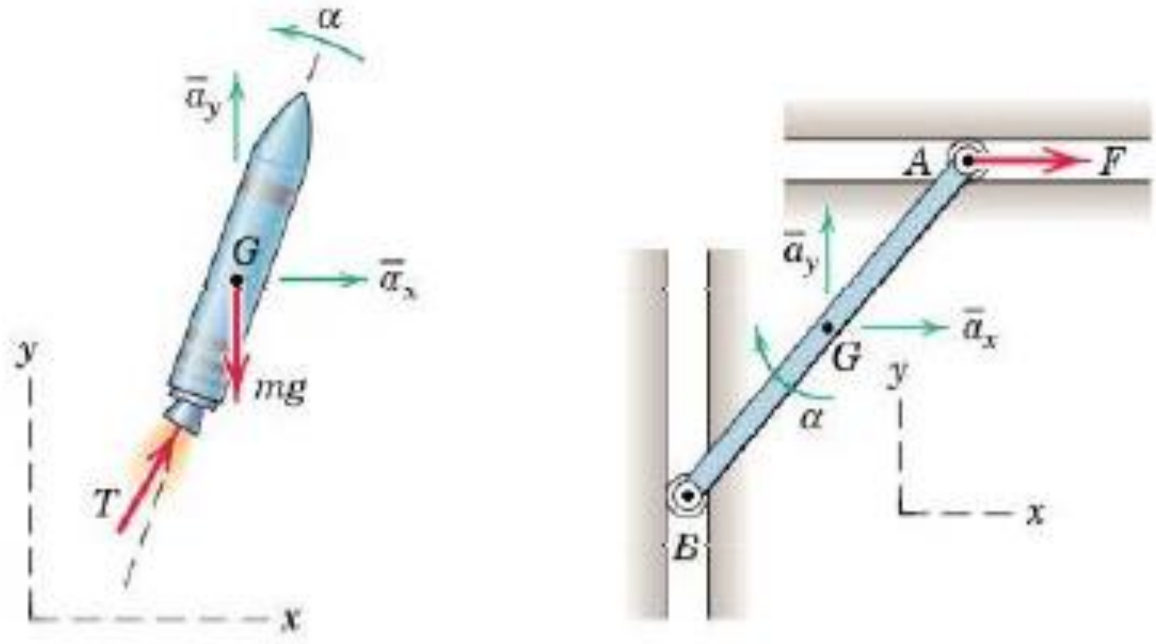
When $\bar{\boldsymbol{\rho}} = \mathbf{0}$, point P becomes the mass center G, and Eq. 6/3 reduces to the scalar form $\Sigma M_G = \bar{I}\alpha$, previously derived. When point P becomes a point O fixed in an inertial reference system and attached to the body (or body extended), then $\mathbf{a}_P = \mathbf{0}$, and Eq. 6/3 in scalar form reduces to

$$\Sigma M_O = I_O \alpha \qquad \textbf{(6/4)}$$

Equation 6/4 then applies to the rotation of a rigid body about a nonaccelerating point O fixed to the body and is the two-dimensional simplification of Eq. 4/7.

Unconstrained and Constrained Motion

The motion of a rigid body may be unconstrained or constrained. The rocket moving in a vertical plane, Fig. 6/6*a*, is an example of unconstrained motion as there are no physical confinements to its motion.



(*a*) Unconstrained Motion (*b*) Constrained Motion

Figure 6/6

The two components $\bar{a}_x$ and $\bar{a}_y$ of the mass-center acceleration and the angular acceleration α may be determined independently of one another by direct application of Eqs. 6/1.

The bar in Fig. 6/6*b*, on the other hand, undergoes a constrained motion, where the vertical and horizontal guides for the ends of the bar impose a kinematic relationship between the acceleration components of the mass center and the angular acceleration of the bar. Thus, it is necessary to determine this kinematic relationship from the principles established in Chapter 5 and to combine it with the force and moment equations of motion before a solution can be carried out.

In general, dynamics problems which involve physical constraints to motion require a kinematic analysis relating linear to angular acceleration before the force and moment equations of motion can be solved. It is for this reason that an understanding of the principles and methods of Chapter 5 is so vital to the work of Chapter 6.

Systems of Interconnected Bodies

Upon occasion, in problems dealing with two or more connected rigid bodies whose motions are related kinematically, it is convenient to analyze the bodies as an entire system.

Figure 6/7 illustrates two rigid bodies hinged at A and subjected to the external forces shown. The forces in the connection at A are internal to the system and are not disclosed. The resultant of all external forces must equal the vector sum of the two resultants $m_1\bar{\mathbf{a}}_1$ and $m_2\bar{\mathbf{a}}_2$, and the sum of the moments about some arbitrary point such as P of all external forces must equal the moment of the resultants, $\bar{I}_1\alpha_1 + \bar{I}_2\alpha_2 + m_1\bar{a}_1d_1 + m_2\bar{a}_2d_2$. Thus, we may state

$$\Sigma\mathbf{F} = \Sigma m\bar{\mathbf{a}}$$
$$\Sigma M_P = \Sigma\bar{I}\alpha + \Sigma m\bar{a}d \qquad (6/5)$$

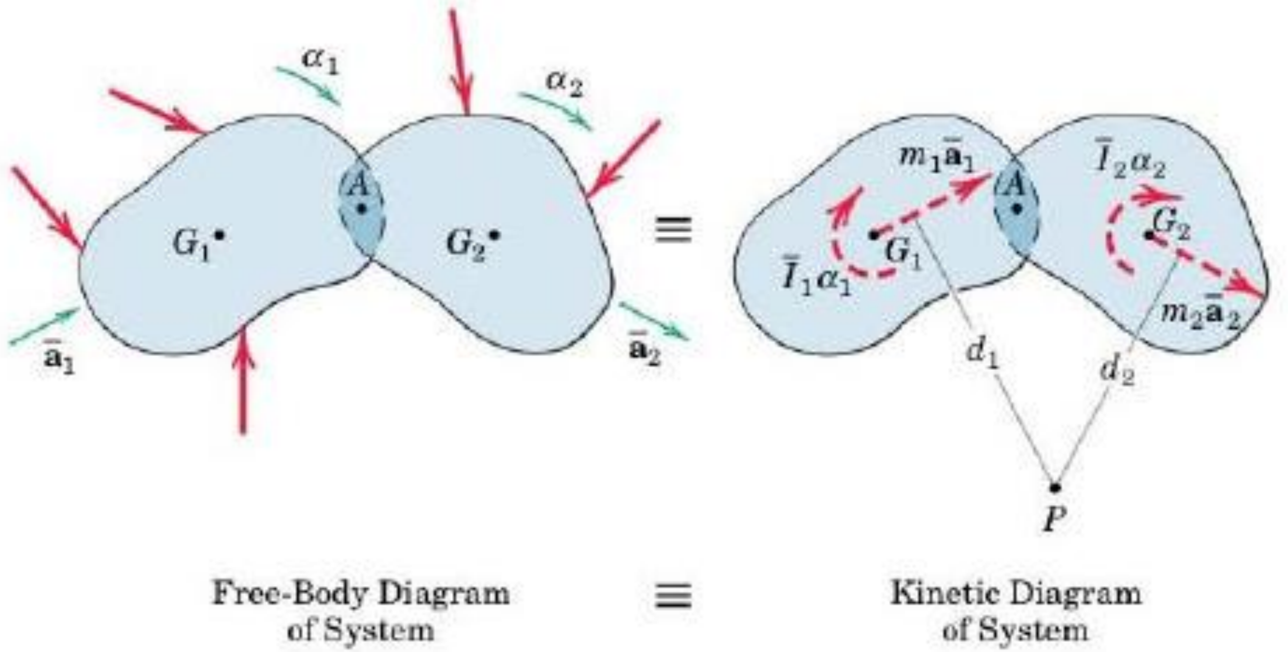


Figure 6/7

where the summations on the right-hand side of the equations represent as many terms as there are separate bodies.

If there are more than three remaining unknowns in a system, however, the three independent scalar equations of motion, when applied to the system, are not sufficient to solve the problem. In this case, more advanced methods such as virtual work (Art. 6/7) or Lagrange's equations (not discussed in this book*) could be employed, or else the system could be dismembered and each part analyzed separately with the resulting equations solved simultaneously.

Analysis Procedure

In the solution of force-mass-acceleration problems for the plane motion of rigid bodies, the following steps should be taken once you understand the conditions and requirements of the problem:

1. Kinematics. First, identify the class of motion and then solve for any needed linear and angular accelerations which can be determined solely from given kinematic information. In the case of constrained plane motion, it is usually necessary to establish the relation between the linear acceleration of the mass center and the angular acceleration of the body by first solving the appropriate relative-velocity and relative-acceleration equations. Again, we emphasize that success in working force-mass-acceleration problems in this chapter is contingent on the ability to describe the necessary kinematics, so that frequent review of Chapter 5 is recommended.

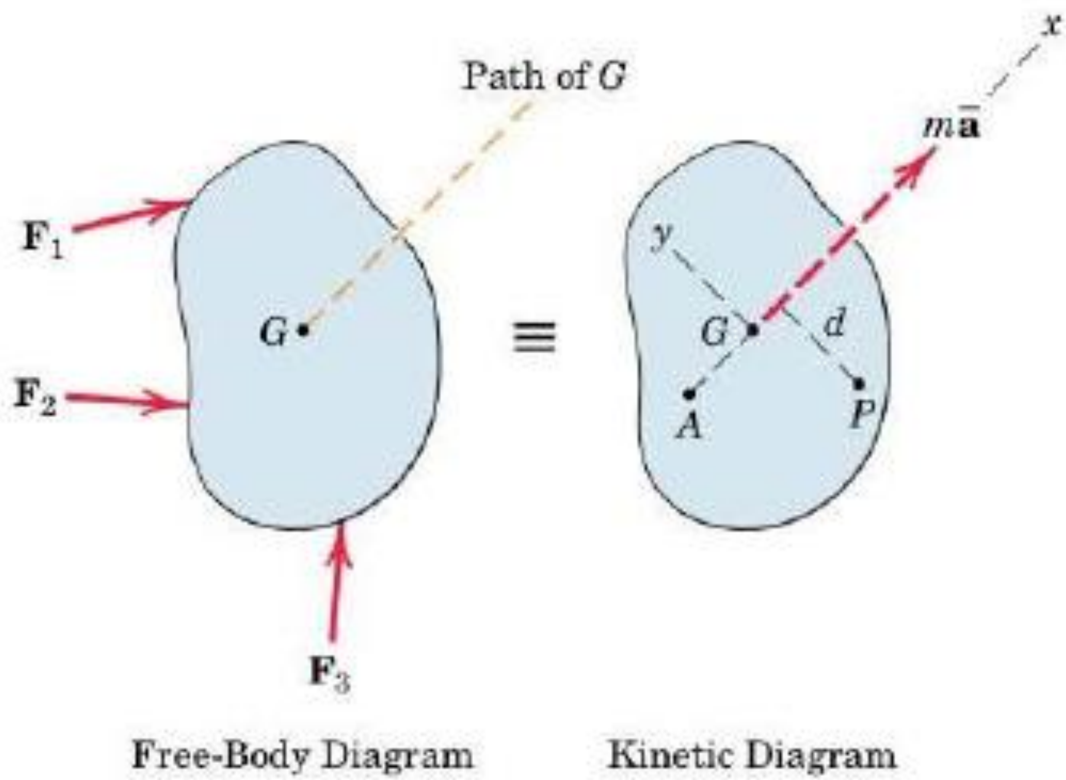
2. Diagrams. Always draw the complete free-body diagram of the body to be analyzed. Assign a convenient inertial coordinate system and label all known and unknown quantities. The kinetic diagram should also be constructed so as to clarify the equivalence between the applied forces and the resulting dynamic response.

3. Equations of Motion. Apply the three equations of motion from Eqs. 6/1, being consistent with the algebraic signs in relation to the choice of reference axes. Equation 6/2 or 6/3 may be employed as an alternative to the second of Eqs. 6/1. Combine these relations with the results from any needed kinematic analysis. Count the number of unknowns and be certain that there are an equal number of independent equations available. For a solvable rigid-body problem in plane motion, there can be no more than the five scalar unknowns which can be determined from the three scalar equations of motion, obtained from Eqs. 6/1, and the two scalar component relations which come from the relative-acceleration equation.

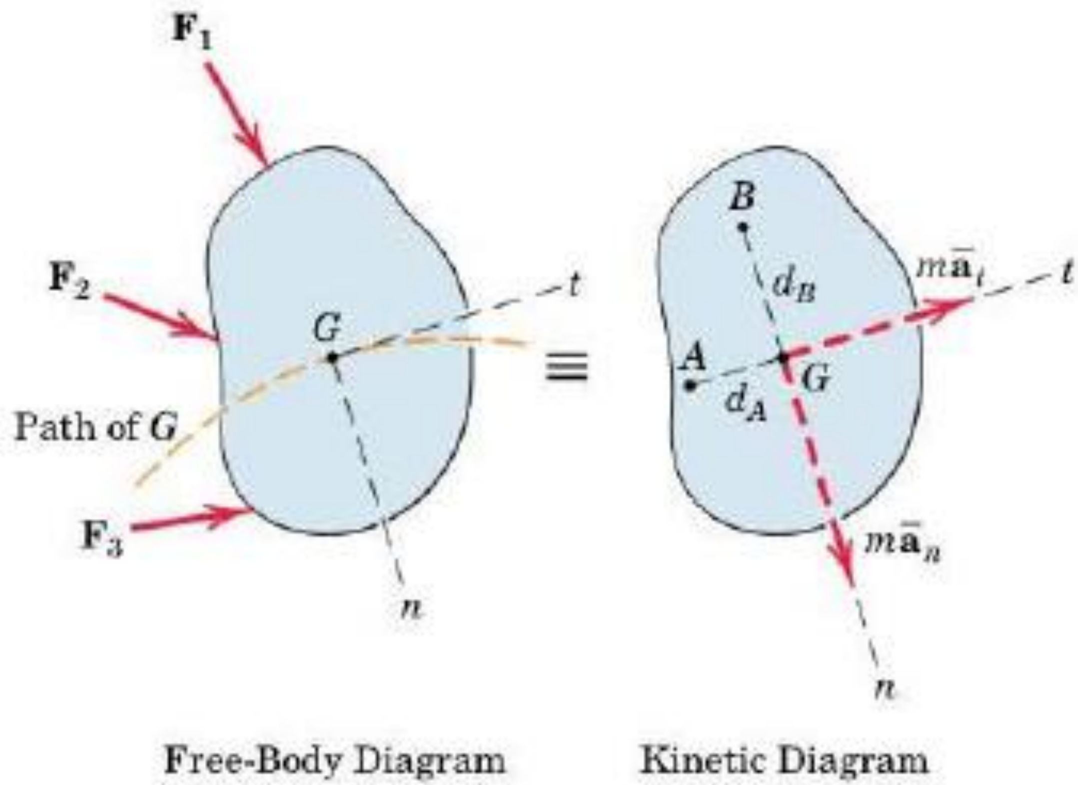
In the following three articles the foregoing developments will be applied to three cases of motion in a plane: *translation*, *fixed-axis rotation*, and *general plane motion*.

6/3 TRANSLATION

Rigid-body translation in plane motion was described in Art. 5/1 and illustrated in Figs. 5/1*a* and 5/1*b*, where we saw that every line in a translating body remains parallel to its original position at all times. In rectilinear translation all points move in straight lines, whereas in curvilinear translation all points move on congruent curved paths. In either case, there is no angular motion of the translating body, so that both ω and α are zero. Therefore, from the moment relation of Eqs. 6/1, we see that all reference to the moment of inertia is eliminated for a translating body.



(*a*) Rectilinear Translation
$(\alpha = 0,\ \omega = 0)$



(b) Curvilinear Translation
($\alpha = 0$, $\omega = 0$)

Figure 6/8

For a translating body, then, our general equations for plane motion, Eqs. 6/1, may be written

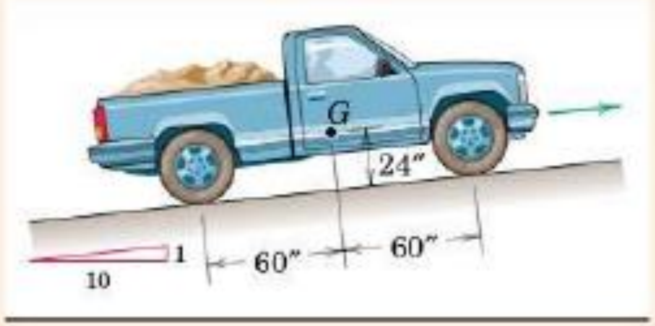
$$\Sigma \mathbf{F} = m\bar{\mathbf{a}}$$
$$\Sigma M_G = \bar{I}\alpha = 0 \qquad (6/6)$$

For rectilinear translation, illustrated in Fig. 6/8*a*, if the *x*-axis is chosen in the direction of the acceleration, then the two scalar force equations become $\Sigma F_x = m\bar{a}_x$ and $\Sigma F_y = m\bar{a}_y = 0$. For curvilinear translation, Fig. 6/8*b*, if we use *n-t* coordinates, the two scalar force equations become $\Sigma F_n = m\bar{a}_n$ and $\Sigma F_t = m\bar{a}_t$. In both cases, $\Sigma M_G = 0$.

We may also employ the alternative moment equation, Eq. 6/2, with the aid of the kinetic diagram. For rectilinear translation we see that $\Sigma M_P = m\bar{a}d$ and $\Sigma M_A = 0$. For curvilinear translation the kinetic diagram permits us to write $\Sigma M_A = m\bar{a}_n d_A$ in the clockwise sense and $\Sigma M_B = m\bar{a}_t d_B$ in the counterclockwise sense. Thus, we have complete freedom to choose a convenient moment center.

Sample Problem 6/1

The pickup truck weighs 3220 lb and reaches a speed of 30 mi/hr from rest in a distance of 200 ft up the 10-percent incline with constant acceleration. Calculate the normal force under each pair of wheels and the friction force under the rear driving wheels. The effective coefficient of friction between the tires and the road is known to be at least 0.8.

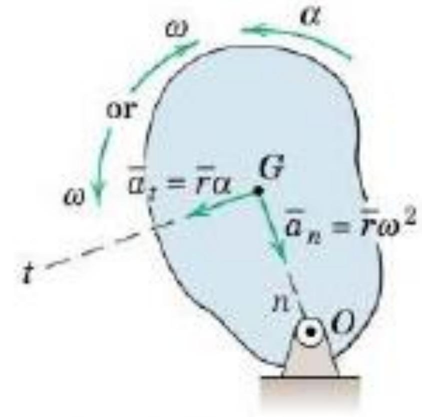


6/4 FIXED-AXIS ROTATION

Rotation of a rigid body about a fixed axis was described under kinematics

For this motion, we saw that all points in the body describe circles about the rotation axis, and all lines of the body in the plane of motion have the same angular velocity ω and angular acceleration α.

The acceleration components of the mass center for circular motion are most easily expressed in n-t coordinates, so we have $a_n = \bar{r}\omega^2$ and $a_t = \bar{r}\alpha$, as shown in Fig. 6/9a for rotation of the rigid body about the fixed axis through O.

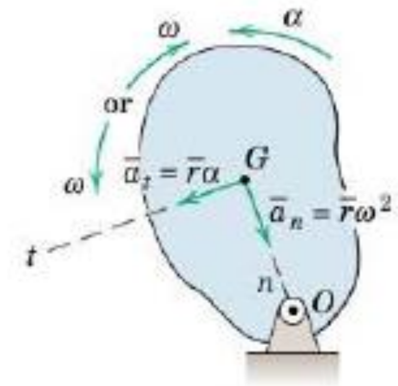


Fixed-Axis Rotation

(a)

Figure 6/9

Part b of the figure represents the free-body diagram, and the equivalent kinetic diagram in part c of the figure shows the force resultant $m\bar{\mathbf{a}}$ in terms of its n- and t-components and the resultant couple $\bar{I}\alpha$.

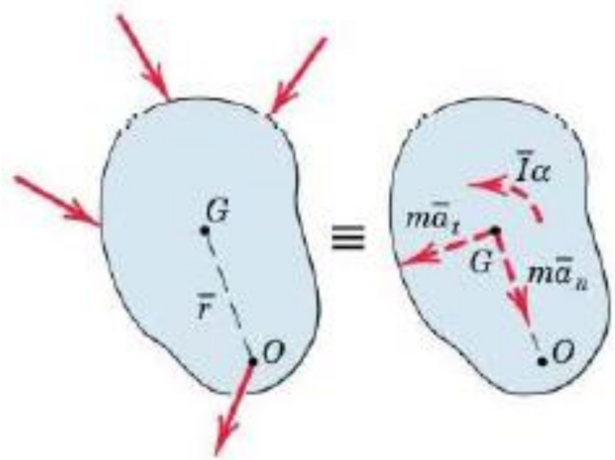


Fixed-Axis Rotation
(a)

Our general equations for plane motion, Eqs. 6/1, are directly applicable and are repeated here.

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}}$$
$$\Sigma M_G = \bar{I}\alpha \qquad [6/1]$$

Thus, the two scalar components of the force equation become $\Sigma F_n = m\bar{r}\omega^2$ and $\Sigma F_t = m\bar{r}\alpha$. In applying the moment equation about G, we must account for the moment of the force applied to the body at O, so this force must not be omitted from the free-body diagram.



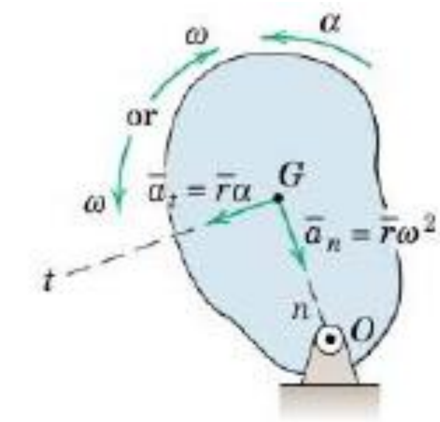
Free-Body Diagram
(b)

Kinetic Diagram
(c)

Figure 6/9

For fixed-axis rotation, it is generally useful to apply a moment equation directly about the rotation axis O. We derived this equation previously as Eq. 6/4, which is repeated here.

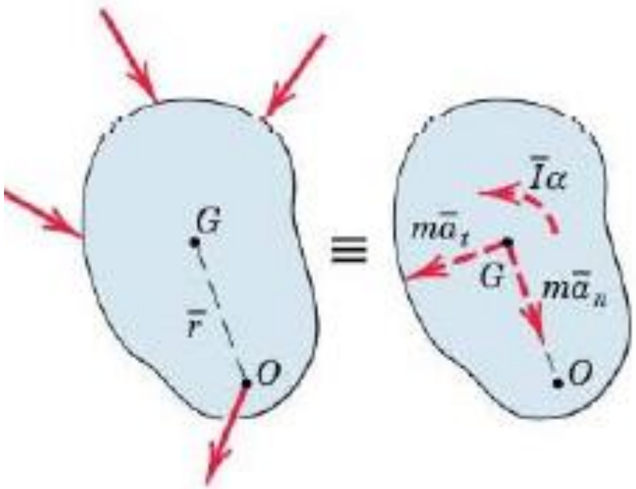
$$\Sigma M_O = I_O\alpha \qquad [6/4]$$



Fixed-Axis Rotation
(a)

From the kinetic diagram in Fig. 6/9c, we may obtain Eq. 6/4 very easily by evaluating the moment of the resultants about O, which becomes $\Sigma M_O = \bar{I}\alpha + m\bar{a}_t\bar{r}$. Application of the parallel-axis theorem for mass moments of inertia, $I_O = \bar{I} + m\bar{r}^2$, gives $\Sigma M_O = (I_O - m\bar{r}^2)\alpha + m\bar{r}^2\alpha = I_O\alpha$.

For the common case of rotation of a rigid body about a fixed axis through its mass center G, clearly, $\bar{\mathbf{a}} = \mathbf{0}$, and therefore $\Sigma\mathbf{F} = \mathbf{0}$. The resultant of the applied forces then is the couple $\bar{I}\alpha$.



Free-Body Diagram
(b)

Kinetic Diagram
(c)

Figure 6/9

We may combine the resultant-force component $m\bar{a}_t$ and resultant couple $\bar{I}\alpha$ by moving $m\bar{a}_t$ to a parallel position through point Q on line OG, Fig. 6/10, located by $m\bar{r}\alpha q = \bar{I}\alpha + m\bar{r}\alpha(\bar{r})$. Using the parallel-axis theorem and $I_O = k_O{}^2 m$ gives $q = k_O{}^2/\bar{r}$.

Point Q is called the *center of percussion* and has the unique property that the resultant of all forces applied to the body must pass through it. It follows that the sum of the moments of all forces about the center of percussion is always zero, $\Sigma M_Q = 0$.

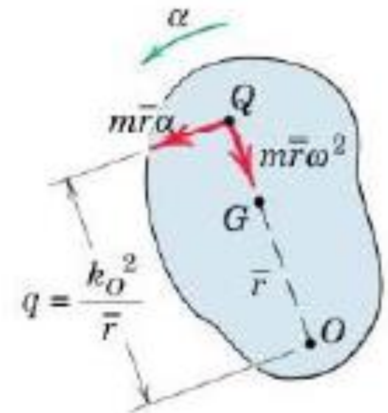
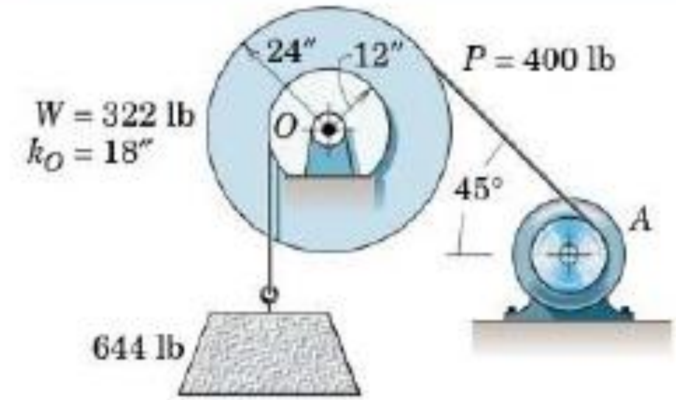


Figure 6/10

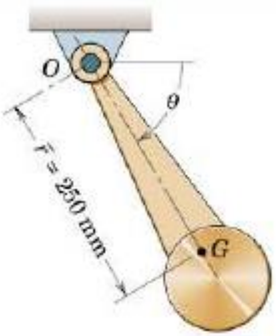
Sample Problem 6/3

The concrete block weighing 644 lb is elevated by the hoisting mechanism shown, where the cables are securely wrapped around the respective drums. The drums, which are fastened together and turn as a single unit about their mass center at O, have a combined weight of 322 lb and a radius of gyration about O of 18 in. If a constant tension P of 400 lb is maintained by the power unit at A, determine the vertical acceleration of the block and the resultant force on the bearing at O.



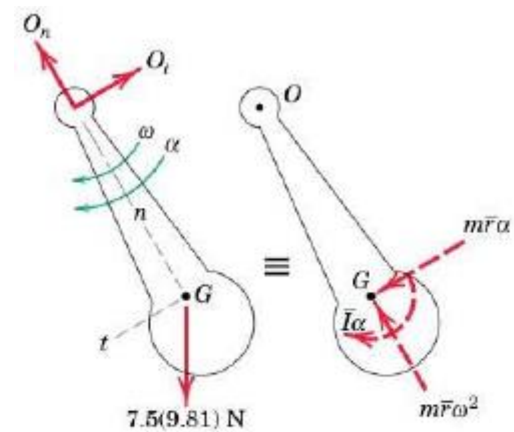
Sample Problem 6/4

The pendulum has a mass of 7.5 kg with center of mass at G and has a radius of gyration about the pivot O of 295 mm. If the pendulum is released from rest at $\theta = 0$, determine the total force supported by the bearing at the instant when $\theta = 60°$. Friction in the bearing is negligible.

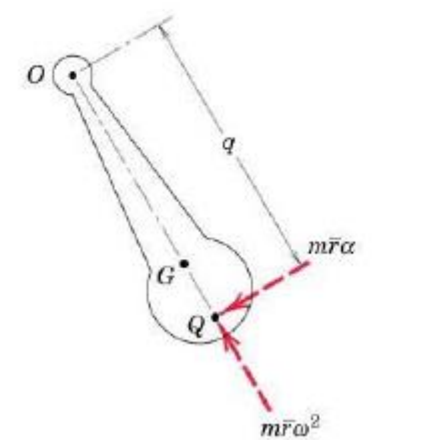


Helpful Hints

① The acceleration components of G are, of course, $\bar{a}_n = \bar{r}\omega^2$ and $\bar{a}_t = \bar{r}\alpha$.



Solution. The free-body diagram of the pendulum in a general position is shown along with the corresponding kinetic diagram, where the components of the resultant force have been drawn through G.



The normal component O_n is found from a force equation in the n-direction, which involves the normal acceleration $\bar{r}\omega^2$. Since the angular velocity ω of the pendulum is found from the integral of the angular acceleration and since O_t depends on the tangential acceleration $\bar{r}\alpha$, it follows that α should be obtained first. To this end with $I_O = k_O{}^2 m$, the moment equation about O gives

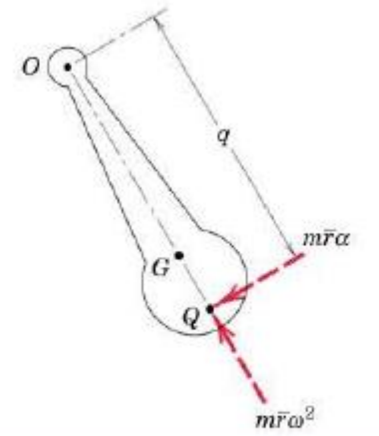
② $[\Sigma M_O = I_O\alpha]$ $\quad 7.5(9.81)(0.25)\cos\theta = (0.295)^2(7.5)\alpha$

$$\alpha = 28.2 \cos\theta \text{ rad/s}^2$$

and for $\theta = 60°$

$[\omega\, d\omega = \alpha\, d\theta]$ $\quad \int_0^{\omega} \omega\, d\omega = \int_0^{\pi/3} 28.2 \cos\theta\, d\theta$

$$\omega^2 = 48.8 \text{ (rad/s)}^2$$



The remaining two equations of motion applied to the 60° position yield

$[\Sigma F_n = m\bar{r}\omega^2]$ $\quad O_n - 7.5(9.81)\sin 60° = 7.5(0.25)(48.8)$

③ $$O_n = 155.2 \text{ N}$$

$[\Sigma F_t = m\bar{r}\alpha]$ $\quad -O_t + 7.5(9.81)\cos 60° = 7.5(0.25)(28.2)\cos 60°$

$$O_t = 10.37 \text{ N}$$

$$O = \sqrt{(155.2)^2 + (10.37)^2} = 155.6 \text{ N} \qquad \textit{Ans.}$$

② Review the theory again and satisfy yourself that $\Sigma M_O = I_O\alpha = \bar{I}\alpha + m\bar{r}^2\alpha = m\bar{r}\alpha q$.

③ Note especially here that the force summations are taken in the positive direction of the acceleration components of the mass center G.

6/5 General Plane Motion

The dynamics of a rigid body in general plane motion combines translation and rotation. In Art. 6/2 we represented such a body in Fig. 6/4 with its free-body diagram and its kinetic diagram, which discloses the dynamic resultants of the applied forces. Figure 6/4 and Eqs. 6/1, which apply to general plane motion, are repeated here for convenient reference.

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}}$$
$$\Sigma M_G = \bar{I}\alpha \qquad \textbf{[6/1]}$$

Direct application of these equations expresses the equivalence between the externally applied forces, as disclosed by the free-body diagram, and their force and moment resultants, as represented by the kinetic diagram.

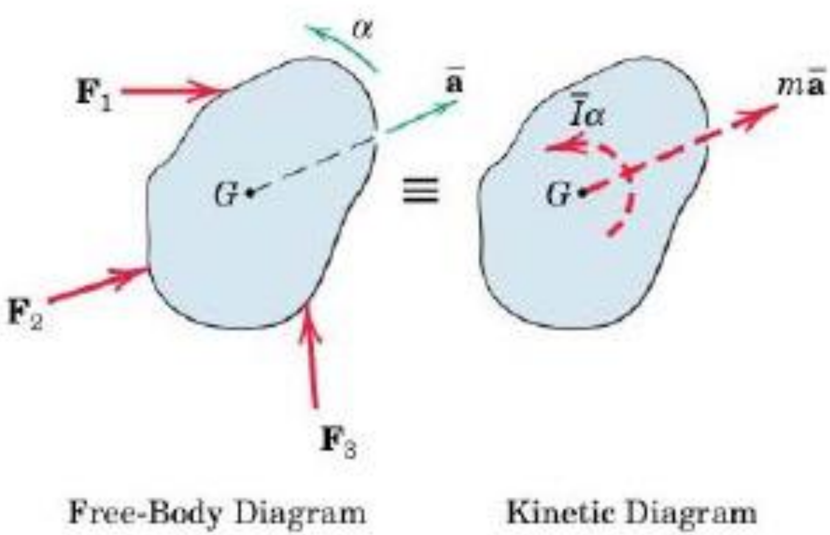


Figure 6/4, repeated

Solving Plane-Motion Problems

Keep in mind the following considerations when solving plane-motion problems.

Choice of Coordinate System. The force equation of Eq. 6/1 should be expressed in whatever coordinate system most readily describes the acceleration of the mass center. You should consider rectangular, normal-tangential, and polar coordinates.

Choice of Moment Equation. In Art. 6/2 we also showed, with the aid of Fig. 6/5, the application of the alternative relation for moments about any point P, Eq. 6/2. This figure and this equation are also repeated here for easy reference.

$$\Sigma M_P = \bar{I}\alpha + m\bar{a}d \qquad [6/2]$$

In some instances, it may be more convenient to use the alternative moment relation of Eq. 6/3 when moments are taken about a point P whose acceleration is known. Note also that the equation for moments about a

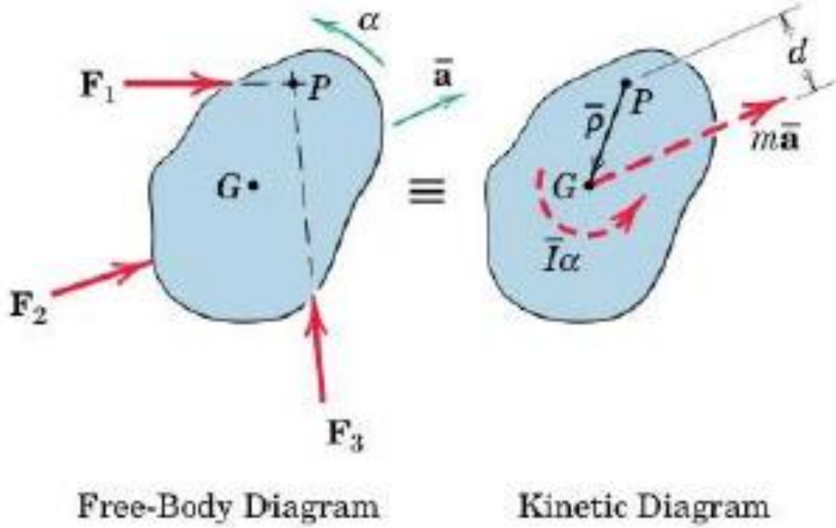


Figure 6/5, repeated

nonaccelerating point O on the body, Eq. 6/4, constitutes still another alternative moment relation and at times may be used to advantage.

Constrained versus Unconstrained Motion. In working a problem in general plane motion, we first observe whether the motion is unconstrained or constrained, as illustrated in the examples of Fig. 6/6. If the motion is constrained, we must account for the kinematic relationship between the linear and the angular accelerations and incorporate it into our force and moment equations of motion. If the motion is unconstrained, the accelerations can be determined independently of one another by direct application of the three motion equations, Eqs. 6/1.

Number of Unknowns. In order for a rigid-body problem to be solvable, the number of unknowns cannot exceed the number of independent equations available to describe them, and a check on the sufficiency of the relationships should always be made. At the most, for plane motion we have three scalar equations of motion and two scalar components of the vector relative-acceleration equation for constrained motion. Thus, we can handle as many as five unknowns for each rigid body.

Identification of the Body or System. We emphasize the importance of clearly choosing the body to be isolated and representing this isolation by a correct free-body diagram. Only after this vital step has been completed can we properly evaluate the equivalence between the external forces and their resultants.

Kinematics. Of equal importance in the analysis of plane motion is a clear understanding of the kinematics involved. Very often, the difficulties experienced at this point have to do with kinematics, and a thorough review of the relative-acceleration relations for plane motion will be most helpful.

Consistency of Assumptions. In formulating the solution to a problem, we recognize that the directions of certain forces or accelerations may not be known at the outset, so that it may be necessary to make initial assumptions whose validity will be proved or disproved when the solution is carried out. It is essential, however, that all assumptions made be consistent with the principle of action and reaction and with any kinematic requirements, which are also called *conditions of constraint*.

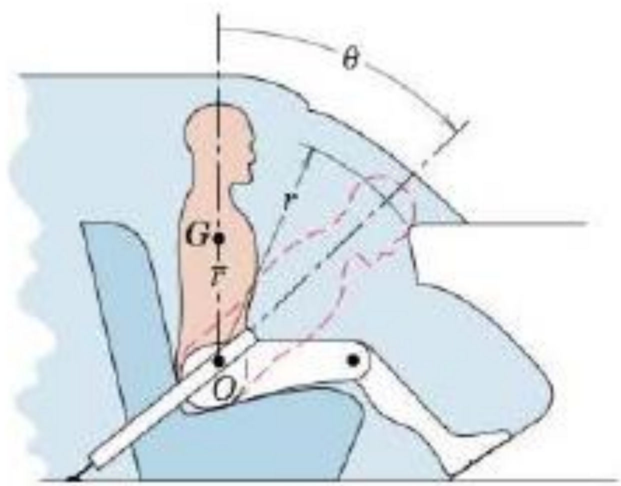
Thus, for example, if a wheel is rolling on a horizontal surface, its center is constrained to move on a horizontal line. Furthermore, if the unknown linear acceleration a of the center of the wheel is assumed positive to the right, then the unknown angular acceleration α will be positive in a clockwise sense in order that $a = +r\alpha$, if we assume the wheel does not slip. Also, we note that, for a wheel which rolls without slipping, the static friction force between the wheel and its supporting surface is generally *less* than its maximum value, so that $F \neq \mu_s N$. But if the wheel slips as it rolls, $a \neq r\alpha$, and a kinetic friction force is generated which is given by $F = \mu_k N$. It may be necessary to test the validity of either assumption, slipping or no slipping, in a given problem. The difference between the coefficients of static and kinetic friction, μ_s and μ_k, is sometimes ignored, in which case, μ is used for either or both coefficients.

Romilly Lockyer/The Image Bank/Getty Images

Look ahead to Prob. 6/107 to see a special-case problem involving a crash-test dummy such as the one shown here.

6/107 In a study of head injury against the instrument panel of a car during sudden or crash stops where lap belts without shoulder straps or airbags are used, the segmented human model shown in the figure is analyzed. The hip joint O is assumed to remain fixed relative to the car, and the torso above the hip is treated as a rigid body of mass m freely pivoted at O. The center of mass of the torso is at G with the initial position of OG taken as vertical. The radius of gyration of the torso about O is k_O. If the car is brought to a sudden stop with a constant deceleration a, determine the velocity v relative to the car with which the model's head strikes the instrument panel. Substitute the values $m = 50$ kg, $\bar{r} = 450$ mm, $r = 800$ mm, $k_O = 550$ mm, $\theta = 45°$, and $a = 10g$ and compute v.

Ans. $v = 11.73$ m/s

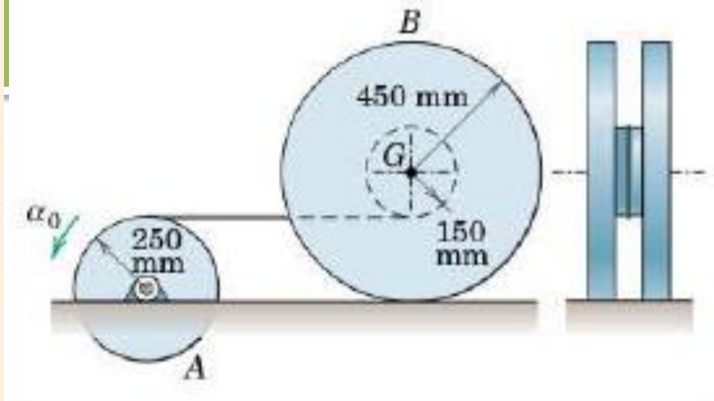


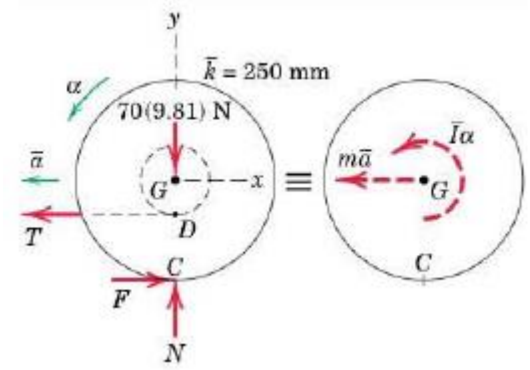
Problem 6/107

Look ahead to Prob. 6/107 to see a special-case problem involving a crash-test dummy such as the one shown here.

Sample Problem 6/6

The drum A is given a constant angular acceleration α_0 of 3 rad/s^2 and causes the 70-kg spool B to roll on the horizontal surface by means of the connecting cable, which wraps around the inner hub of the spool. The radius of gyration $\bar{k}$ of the spool about its mass center G is 250 mm, and the coefficient of static friction between the spool and the horizontal surface is 0.25. Determine the tension T in the cable and the friction force F exerted by the horizontal surface on the spool.





Solution. The free-body diagram and the kinetic diagram of the spool are drawn as shown. The correct direction of the friction force may be assigned in this problem by observing from both diagrams that with counterclockwise angular acceleration, a moment sum about point G (and also about point D) must be counterclockwise. A point on the connecting cable has an acceleration $a_t = r\alpha = 0.25(3) = 0.75$ m/s^2, which is also the horizontal component of the acceleration of point D on the spool. It will be assumed initially that the spool rolls without slipping, in which case it has a counterclockwise angular acceleration $\alpha =$
① $(a_D)_x/\overline{DC} = 0.75/0.30 = 2.5$ rad/s^2. The acceleration of the mass center G is, therefore, $\bar{a} = r\alpha = 0.45(2.5) = 1.125$ m/s^2.

With the kinematics determined, we now apply the three equations of motion, Eqs. 6/1,

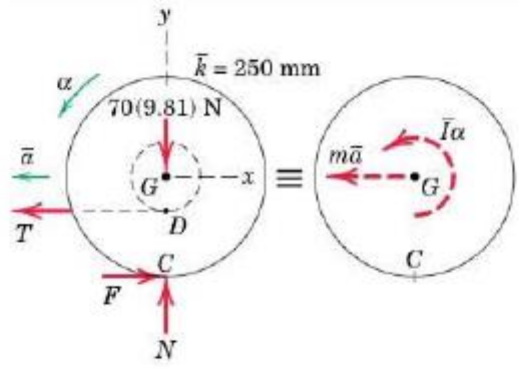
$[\Sigma F_x = m\bar{a}_x]$ $\quad F - T = 70(-1.125) \quad (a)$

$[\Sigma F_y = m\bar{a}_y]$ $\quad N - 70(9.81) = 0 \qquad N = 687$ N

② $[\Sigma M_G = \bar{I}\alpha]$ $\quad F(0.450) - T(0.150) = 70(0.250)^2(2.5) \quad (b)$

Solving (a) and (b) simultaneously gives

$F = 75.8$ N and $T = 154.6$ N *Ans.*



Helpful Hints

① The relation between $\bar{a}$ and α is the kinematic constraint which accompanies the assumption that the spool rolls without slipping.

② Be careful not to make the mistake of using $\frac{1}{2}mr^2$ for $\bar{I}$ of the spool, which is not a uniform circular disk.

To establish the validity of our assumption of no slipping, we see that the surfaces are capable of supporting a maximum friction force $F_{max} = \mu_s N = 0.25(687) = 171.7$ N. Since only 75.8 N of friction force is required, we conclude that our assumption of rolling without slipping is valid.

If the coefficient of static friction had been 0.1, for example, then the friction force would have been limited to $0.1(687) = 68.7$ N, which is less than 75.8 N, and the spool would slip. In this event, the kinematic relation $\bar{a} = r\alpha$ would
③ no longer hold. With $(a_D)_x$ known, the angular acceleration would be $\alpha = [\bar{a} - (a_D)_x]/\overline{GD}$. Using this relation along with $F = \mu_k N = 68.7$ N, we would then resolve the three equations of motion for the unknowns T, $\bar{a}$, and α.

Alternatively, with point C as a moment center in the case of pure rolling, we may use Eq. 6/2 and obtain T directly. Thus,

$$[\Sigma M_C = \bar{I}\alpha + m\bar{a}r] \quad 0.3T = 70(0.25)^2(2.5) + 70(1.125)(0.45)$$

$$T = 154.6 \text{ N} \qquad \textit{Ans.}$$

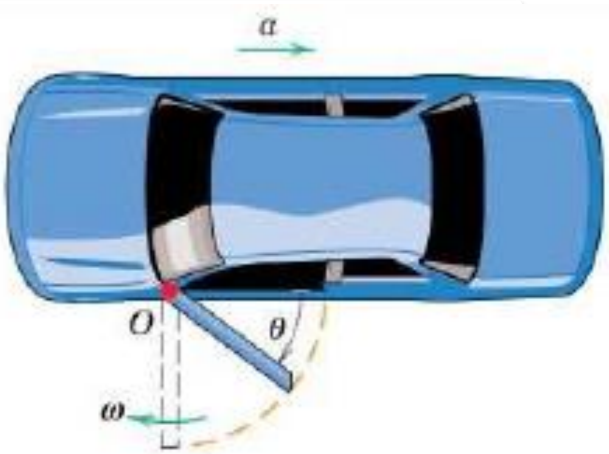
where the previous kinematic results for no slipping have been incorporated. We
④ could also write a moment equation about point D to obtain F directly.

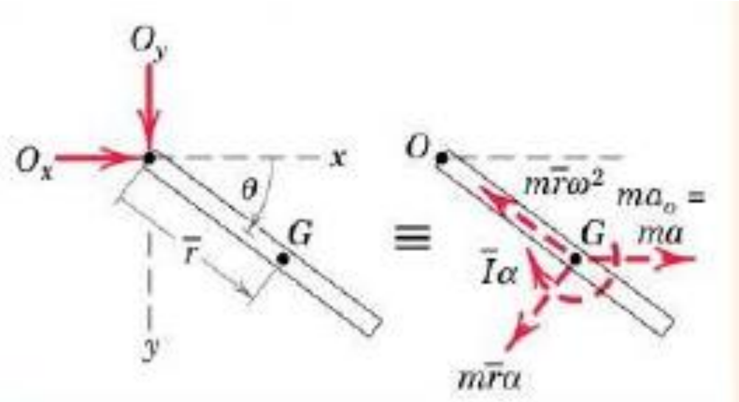
③ Our principles of relative acceleration are a necessity here. Hence, the relation $(a_{G/D})_t = \overline{GD}\alpha$ should be recognized.

④ The flexibility in the choice of moment centers provided by the kinetic diagram can usually be employed to simplify the analysis.

Sample Problem 6/8

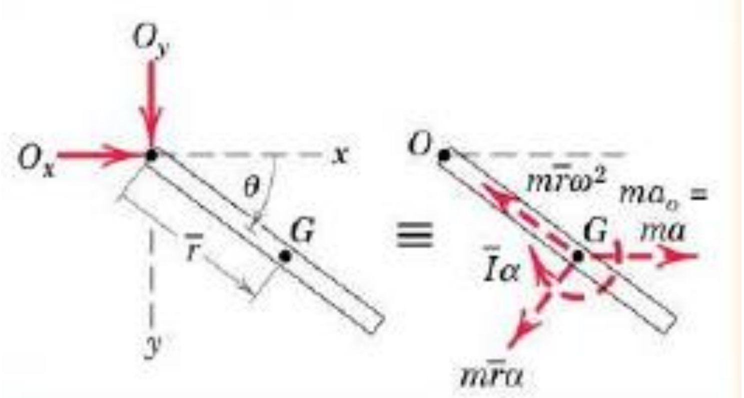
A car door is inadvertently left slightly open when the brakes are applied to give the car a constant rearward acceleration a. Derive expressions for the angular velocity of the door as it swings past the 90° position and the components of the hinge reactions for any value of θ. The mass of the door is m, its mass center is a distance $\bar{r}$ from the hinge axis O, and the radius of gyration about O is k_O.





Helpful Hints

① Point O is chosen because it is the only point on the door whose acceleration is known.



Solution. Because the angular velocity ω increases with θ, we need to find how the angular acceleration α varies with θ so that we may integrate it over the interval to obtain ω. We obtain α from a moment equation about O. First, we draw the free-body diagram of the door in the horizontal plane for a general position θ. The only forces in this plane are the components of the hinge reaction shown here in the x- and y-directions. On the kinetic diagram, in addition to the resultant couple $\bar{I}\alpha$ shown in the sense of α, we represent the resultant force $m\bar{\mathbf{a}}$ in terms of its components by using an equation of relative acceleration with respect to O. This equation becomes the kinematic equation of constraint and is ①

$$\bar{\mathbf{a}} = \mathbf{a}_G = \mathbf{a}_O + (\mathbf{a}_{G/O})_n + (\mathbf{a}_{G/O})_t$$

The magnitudes of the $m\bar{\mathbf{a}}$ components are then

② $$ma_O = ma \qquad m(a_{G/O})_n = m\bar{r}\omega^2 \qquad m(a_{G/O})_t = m\bar{r}\alpha$$

where $\omega = \dot{\theta}$ and $\alpha = \ddot{\theta}$.

② Be careful to place $m\bar{r}\alpha$ in the sense of positive α with respect to rotation about O.

For a given angle θ, the three unknowns are α, O_x, and O_y. We can eliminate O_x and O_y by a moment equation about O, which gives

③ $[\Sigma M_O = \bar{I}\alpha + \Sigma m\bar{a}d]$ $\qquad 0 = m(k_O{}^2 - \bar{r}^2)\alpha + m\bar{r}\alpha(\bar{r}) - ma(\bar{r}\sin\theta)$

④ Solving for α gives $\qquad \alpha = \frac{a\bar{r}}{k_O{}^2}\sin\theta$

Now we integrate α first to a general position and get

$[\omega\, d\omega = \alpha\, d\theta]$ $\qquad \int_0^\omega \omega\, d\omega = \int_0^\theta \frac{a\bar{r}}{k_O{}^2}\sin\theta\, d\theta$

$$\omega^2 = \frac{2a\bar{r}}{k_O{}^2}(1 - \cos\theta)$$

For $\theta = \pi/2$, $\qquad \omega = \frac{1}{k_O}\sqrt{2a\bar{r}}$ *Ans.*

③ The free-body diagram shows that there is zero moment about O. We use the transfer-of-axis theorem here and substitute $k_O{}^2 = \bar{k}^2 + \bar{r}^2$. If this relation is not totally familiar, review Art. B/1 in Appendix B.

④ We may also use Eq. 6/3 with O as a moment center

$$\Sigma \mathbf{M}_O = I_O\boldsymbol{\alpha} + \bar{\boldsymbol{\rho}} \times m\mathbf{a}_O$$

where the scalar values of the terms are $I_O\alpha = mk_O{}^2\alpha$ and $\bar{\boldsymbol{\rho}} \times m\mathbf{a}_O$ becomes $-\bar{r}ma\sin\theta$.

⑤ The kinetic diagram shows clearly the terms which make up $m\bar{a}_x$ and $m\bar{a}_y$.

⑤ To find O_x and O_y for any given value of θ, the force equations give

$$[\Sigma F_x = m\bar{a}_x] \qquad O_x = ma - m\bar{r}\omega^2 \cos\theta - m\bar{r}\alpha \sin\theta$$

$$= m\left[a - \frac{2a\bar{r}^2}{k_O^2}(1 - \cos\theta)\cos\theta - \frac{a\bar{r}^2}{k_O^2}\sin^2\theta\right]$$

$$= ma\left[1 - \frac{\bar{r}^2}{k_O^2}(1 + 2\cos\theta - 3\cos^2\theta)\right] \qquad \textit{Ans.}$$

$$[\Sigma F_y = m\bar{a}_y] \qquad O_y = m\bar{r}\alpha \cos\theta - m\bar{r}\omega^2 \sin\theta$$

$$= m\bar{r}\frac{a\bar{r}}{k_O^2}\sin\theta\cos\theta - m\bar{r}\frac{2a\bar{r}}{k_O^2}(1 - \cos\theta)\sin\theta$$

$$= \frac{ma\bar{r}^2}{k_O^2}(3\cos\theta - 2)\sin\theta \qquad \textit{Ans.}$$

SECTION B. WORK AND ENERGY

Work of Forces and Couples

The work done by a force **F** has been treated in detail in Art. 3/6 and is given by

$$U = \int \mathbf{F} \cdot d\mathbf{r} \qquad \text{or} \qquad U = \int (F \cos \alpha)\, ds$$

We frequently need to evaluate the work done by a couple M which acts on a rigid body during its motion. Figure 6/11 shows a couple $M = Fb$ acting on a rigid body which moves in the plane of the couple. During time dt the body rotates through an angle $d\theta$, and line AB moves to $A'B'$. We may consider this motion in two parts, first a translation to $A'B''$ and then a rotation $d\theta$ about A'. We see immediately that during the translation the work done by one of the forces cancels that done by the other force, so that the net work done is $dU = F(b\, d\theta) = M\, d\theta$ due to the rotational part of the motion. If the couple acts in the sense opposite to the rotation, the work done is negative. During a finite rotation, the work done by a couple M whose plane is parallel to the plane of motion is, therefore,

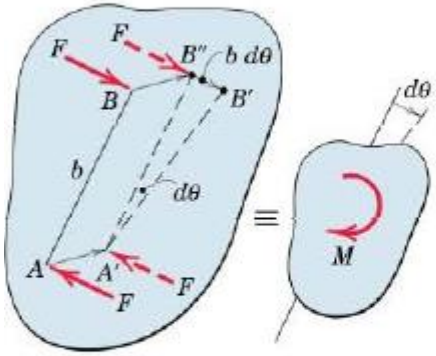


Figure 6/11

$$U = \int M\, d\theta$$

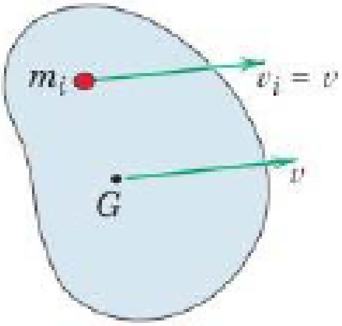
Kinetic Energy

We now use the familiar expression for the kinetic energy of a particle to develop expressions for the kinetic energy of a rigid body for each of the three classes of rigid-body plane motion illustrated in Fig. 6/12.

(a) Translation. The translating rigid body of Fig. 6/12*a* has a mass m and all of its particles have a common velocity v. The kinetic energy of any particle of mass m_i of the body is $T_i = \frac{1}{2} m_i v^2$, so for the entire body $T = \Sigma \frac{1}{2} m_i v^2 = \frac{1}{2} v^2 \Sigma m_i$ or

$$T = \frac{1}{2} m v^2 \qquad \textbf{(6/7)}$$

This expression holds for both rectilinear and curvilinear translation.

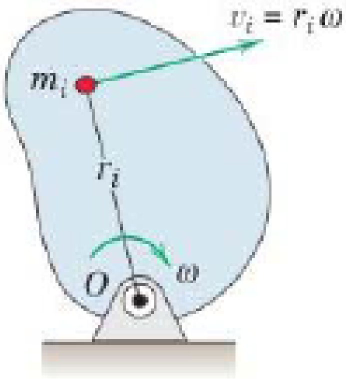


(*a*) Translation

(b) Fixed-axis rotation. The rigid body in Fig. 6/12*b* rotates with an angular velocity ω about the fixed axis through O. The kinetic energy of a representative particle of mass m_i is $T_i = \frac{1}{2} m_i(r_i\omega)^2$. Thus, for the entire body $T = \frac{1}{2}\omega^2 \Sigma m_i r_i^2$. But the moment of inertia of the body about O is $I_O = \Sigma m_i r_i^2$, so

$$T = \frac{1}{2} I_O \omega^2 \qquad \textbf{(6/8)}$$

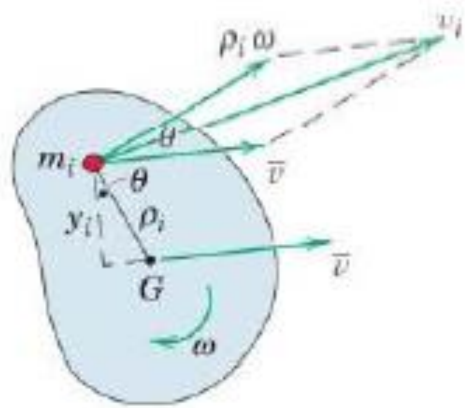
Note the similarity in the forms of the kinetic energy expressions for translation and rotation. You should verify that the dimensions of the two expressions are identical.



(*b*) Fixed-Axis Rotation

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2 \qquad \textbf{(6/9)}$$

where $\bar{I}$ is the moment of inertia of the body about its mass center. This expression for kinetic energy clearly shows the separate contributions to the total kinetic energy resulting from the translational velocity $\bar{v}$ of the mass center and the rotational velocity ω about the mass center.



(c) General Plane Motion

Figure 6/12

The kinetic energy of plane motion may also be expressed in terms of the rotational velocity about the instantaneous center C of zero velocity. Because C momentarily has zero velocity, the proof leading to Eq. 6/8 for the fixed point O holds equally well for point C, so that, alternatively, we may write the kinetic energy of a rigid body in plane motion as

$$T = \frac{1}{2} I_C \omega^2 \qquad \textbf{(6/10)}$$

Potential Energy and the Work-Energy Equation

Gravitational potential energy V_g and elastic potential energy V_e were covered in detail in Art. 3/7. Recall that the symbol U' (rather than U) is used to denote the work done by all forces except the weight and elastic forces, which are accounted for in the potential-energy terms.

$$T_1 + U_{1\text{-}2} = T_2 \qquad [4/2]$$

applies to any mechanical system. For application to the motion of a single rigid body, the terms T_1 and T_2 must include the effects of translaton and rotation as given by Eqs. 6/7, 6/8, 6/9, or 6/10, and $U_{1\text{-}2}$ is the work done by all external forces. On the other hand, if we choose to express the effects of weight and springs by means of potential energy rather than work, we may rewrite the above equation as

$$T_1 + V_1 + U'_{1\text{-}2} = T_2 + V_2 \qquad [4/3a]$$

where the prime denotes the work done by all forces other than weight and spring forces.

Power

The concept of power was discussed in Art. 3/6, which treated work-energy for particle motion. Recall that power is the time rate at which work is performed. For a force **F** acting on a rigid body in plane motion, the power developed by that force at a given instant is given by Eq. 3/16 and is the rate at which the force is doing work. The power is given by

$$P = \frac{dU}{dt} = \frac{\mathbf{F}\cdot d\mathbf{r}}{dt} = \mathbf{F}\cdot\mathbf{v}$$

where $d\mathbf{r}$ and $\mathbf{v}$ are, respectively, the differential displacement and the velocity of the point of application of the force.

Similarly, for a couple M acting on the body, the power developed by the couple at a given instant is the rate at which it is doing work, and is given by

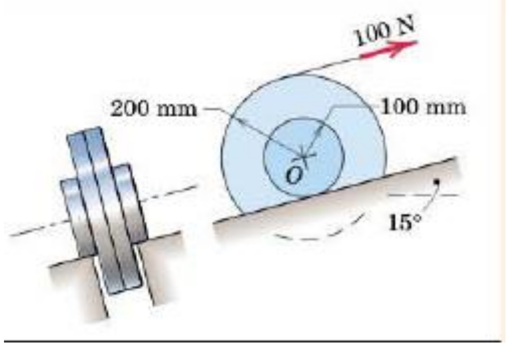
$$P = \frac{dU}{dt} = \frac{M\,d\theta}{dt} = M\omega$$

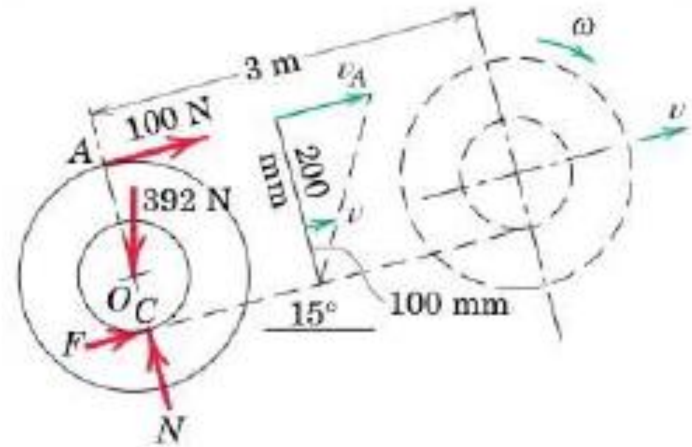
where $d\theta$ and ω are, respectively, the differential angular displacement and the angular velocity of the body. If the senses of M and ω are the same, the power is positive and energy is supplied to the body. Conversely, if M and ω have opposite senses, the power is negative and energy is removed from the body. If the force $\mathbf{F}$ and the couple M act simultaneously, the total instantaneous power is

$$P = \mathbf{F}\cdot\mathbf{v} + M\omega$$

Sample Problem 6/9

The wheel rolls up the incline on its hubs without slipping and is pulled by the 100-N force applied to the cord wrapped around its outer rim. If the wheel starts from rest, compute its angular velocity ω after its center has moved a distance of 3 m up the incline. The wheel has a mass of 40 kg with center of mass at O and has a centroidal radius of gyration of 150 mm. Determine the power input from the 100-N force at the end of the 3-m motion interval.





Solution. Of the four forces shown on the free-body diagram of the wheel, only ① the 100-N pull and the weight of 40(9.81) = 392 N do work. The friction force does no work as long as the wheel does not slip. By use of the concept of the instantaneous center C of zero velocity, we see that a point A on the cord to which the 100-N force is applied has a velocity $v_A = [(200 + 100)/100]v$. Hence, point A on the cord moves a distance of (200 + 100)/100 = 3 times as far as the center O. Thus, with the effect of the weight included in the U-term, the work done on the wheel becomes

② $$U_{1\text{-}2} = 100\,\frac{200 + 100}{100}(3) - (392 \sin 15°)(3) = 595 \text{ J}$$

The wheel has general plane motion, so that the initial and final kinetic energies are

③ $[T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2]$ $T_1 = 0$ $T_2 = \frac{1}{2}40(0.10\omega)^2 + \frac{1}{2}40(0.15)^2\omega^2$

$$= 0.650\omega^2$$

The work-energy equation gives

$[T_1 + U_{1\text{-}2} = T_2]$ $0 + 595 = 0.650\omega^2$ $\omega = 30.3$ rad/s

Alternatively, the kinetic energy of the wheel may be written

④ $[T = \frac{1}{2}I_C\omega^2]$ $T = \frac{1}{2}40[(0.15)^2 + (0.10)^2]\omega^2 = 0.650\omega^2$

The power input from the 100-N force when $\omega = 30.3$ rad/s is

⑤ $[P = \mathbf{F}\cdot\mathbf{v}]$ $P_{100} = 100(0.3)(30.3) = 908$ W *Ans.*

Helpful Hints

① Since the velocity of the instantaneous center C on the wheel is zero, it follows that the rate at which the friction force does work is continuously zero. Hence, F does no work as long as the wheel does not slip. If the wheel were rolling on a moving platform, however, the friction force would do work, even if the wheel were not slipping.

② Note that the component of the weight down the plane does negative work.

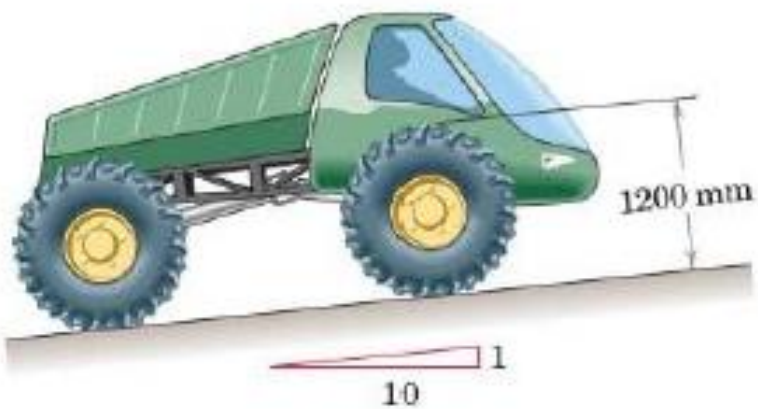
③ Be careful to use the correct radius in the expression $v = r\omega$ for the velocity of the center of the wheel.

④ Recall that $I_C = \bar{I} + m\overline{OC}^2$, where $\bar{I} = I_O = mk_O{}^2$.

⑤ The velocity here is that of the application point of the 100-N force.

6/145 A small experimental vehicle has a total mass m of 500 kg including wheels and driver. Each of the four wheels has a mass of 40 kg and a centroidal radius of gyration of 400 mm. Total frictional resistance R to motion is 400 N and is measured by towing the vehicle at a constant speed on a level road with engine disengaged. Determine the power output of the engine for a speed of 72 km/h up the 10-percent grade (*a*) with zero acceleration and (*b*) with an acceleration of 3 m/s^2. (*Hint:* Power equals the time rate of increase of the total energy of the vehicle plus the rate at which frictional work is overcome.)

Ans. (*a*) $P = 17.76$ kW, (*b*) $P = 52.0$ kW



Problem 6/145

6/7 Acceleration from Work-Energy; Virtual Work

In addition to using the work-energy equation to determine the velocities due to the action of forces acting over finite displacements, we may also use the equation to establish the instantaneous accelerations of the members of a system of interconnected bodies as a result of the active forces applied. We may also modify the equation to determine the configuration of such a system when it undergoes a constant acceleration.

Work-Energy Equation for Differential Motions

For an infinitesimal interval of motion, Eq. 4/3 becomes

$$dU' = dT + dV$$

The term dU' represents the total work done by all active nonpotential forces acting on the system under consideration during the infinitesimal displacement of the system. The work of potential forces is included in the dV-term. If we use the subscript i to denote a representative body of the interconnected system, the differential change in kinetic energy T for the entire system becomes

$$dT = d(\Sigma\tfrac{1}{2}m_i\bar{v}_i^{\,2} + \Sigma\tfrac{1}{2}\bar{I}_i\omega_i^{\,2}) = \Sigma m_i\bar{v}_i\,d\bar{v}_i + \Sigma\bar{I}_i\omega_i\,d\omega_i$$

where $d\bar{v}_i$ and $d\omega_i$ are the respective changes in the magnitudes of the velocities and where the summation is taken over all bodies of the system. But for each body, $m_i\bar{v}_i\,d\bar{v}_i = m_i\bar{\mathbf{a}}_i\cdot d\bar{\mathbf{s}}_i$ and $\bar{I}_i\omega_i\,d\omega_i = \bar{I}_i\alpha_i\,d\theta_i$, where $d\bar{\mathbf{s}}_i$ represents the infinitesimal linear displacement of the center of mass and where $d\theta_i$ represents the infinitesimal angular displacement of the body in the plane of motion. We note that $\bar{\mathbf{a}}_i\cdot d\bar{\mathbf{s}}_i$ is identical to $(\bar{a}_i)_t\,d\bar{s}_i$, where $(\bar{a}_i)_t$ is the component of $\bar{\mathbf{a}}_i$ along the tangent to the curve described by the mass center of the body in question. Also α_i represents $\ddot{\theta}_i$, the angular acceleration of the representative body. Consequently, for the entire system

$$dT = \Sigma m_i\bar{\mathbf{a}}_i\cdot d\bar{\mathbf{s}}_i + \Sigma\bar{I}_i\alpha_i\,d\theta_i$$

This change may also be written as

$$dT = \Sigma \mathbf{R}_i \cdot d\bar{\mathbf{s}}_i + \Sigma \mathbf{M}_{G_i} \cdot d\boldsymbol{\theta}_i$$

where $\mathbf{R}_i$ and $\mathbf{M}_{G_i}$ are the resultant force and resultant couple acting on body i and where $d\boldsymbol{\theta}_i = d\theta_i \mathbf{k}$. These last two equations merely show us that the differential change in kinetic energy equals the differential work done on the system by the resultant forces and resultant couples acting on all the bodies of the system.

The term dV represents the differential change in the total gravitational potential energy V_g and the total elastic potential energy V_e and has the form

$$dV = d(\Sigma m_i g h_i + \Sigma \frac{1}{2} k_j x_j^2) = \Sigma m_i g \, dh_i + \Sigma k_j x_j \, dx_j$$

where h_i represents the vertical distance of the center of mass of the representative body of mass m_i above any convenient datum plane and where x_j stands for the deformation, tensile or compressive, of a representative elastic member of the system (spring) whose stiffness is k_j.

The complete expression for dU' may now be written as

$$dU' = \Sigma m_i \bar{\mathbf{a}}_i \cdot d\bar{\mathbf{s}}_i + \Sigma \bar{I}_i \alpha_i \, d\theta_i + \Sigma m_i g \, dh_i + \Sigma k_j x_j \, dx_j \qquad \textbf{(6/11)}$$

Virtual Work

In Eq. 6/11 the differential motions are differential changes in the real or actual displacements which occur. For a mechanical system which assumes a steady-state configuration during constant acceleration, we often find it convenient to introduce the concept of *virtual work*. The concepts of virtual work and virtual displacement were introduced and used to establish equilibrium configurations for static systems of interconnected bodies (see Chapter 7 of *Vol. 1 Statics*).

A *virtual displacement* is any assumed and arbitrary displacement, linear or angular, away from the natural or actual position. For a system of connected bodies, the virtual displacements must be consistent with the constraints of the system. For example, when one end of a link is hinged about a fixed pivot, the virtual displacement of the other end must be normal to the line joining the two ends. Such requirements for displacements consistent with the constraints are purely kinematic and provide what are known as the *equations of constraint*.

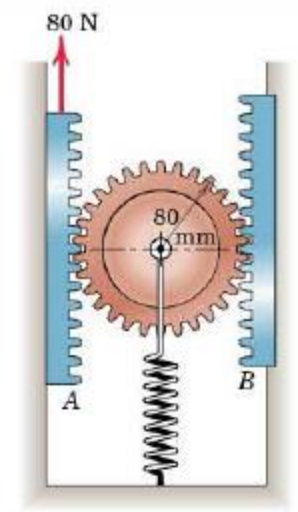
If a set of virtual displacements satisfying the equations of constraint and therefore consistent with the constraints is assumed for a mechanical system, the proper relationship between the coordinates which specify the configuration of the system will be determined by applying the work-energy relationship of Eq. 6/11, expressed in terms of virtual changes. Thus,

$$\delta U' = \Sigma m_i \bar{\mathbf{a}}_i \cdot \delta \bar{\mathbf{s}}_i + \Sigma \bar{I}_i \alpha_i \, \delta\theta_i + \Sigma m_i g \, \delta h_i + \Sigma k_j x_j \, \delta x_j \qquad \textbf{(6/11a)}$$

It is customary to use the differential symbol d to refer to differential changes in the *real* displacements, whereas the symbol δ is used to signify virtual changes, that is, differential changes which are *assumed* rather than real.

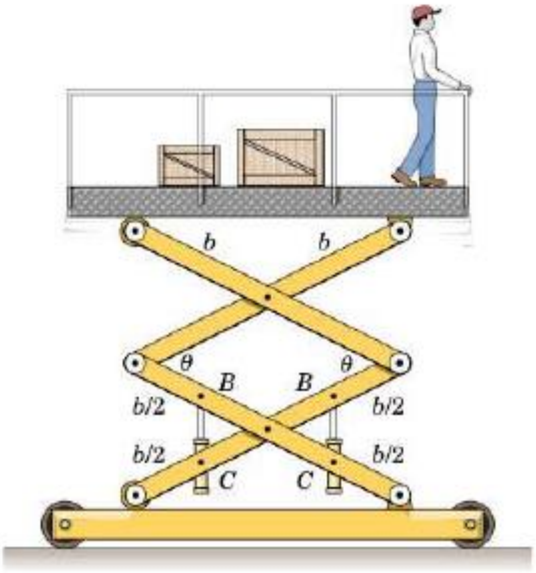
Sample Problem 6/12

The movable rack A has a mass of 3 kg, and rack B is fixed. The gear has a mass of 2 kg and a radius of gyration of 60 mm. In the position shown, the spring, which has a stiffness of 1.2 kN/m, is stretched a distance of 40 mm. For the instant represented, determine the acceleration a of rack A under the action of the 80-N force. The plane of the figure is vertical.



6/165 The portable work platform is elevated by means of the two hydraulic cylinders articulated at points C. The pressure in each cylinder produces a force F. The platform, man, and load have a combined mass m, and the mass of the linkage is small and may be neglected. Determine the upward acceleration a of the platform and show that it is independent of both b and θ.

Ans. $a = \frac{F}{2m} - g$



Problem 6/165

SECTION C. IMPULSE AND MOMENTUM

6/8 IMPULSE-MOMENTUM EQUATIONS

Linear Momentum

$\mathbf{G} = d(m\bar{\mathbf{r}})/dt = m\dot{\bar{\mathbf{r}}}$, where $\dot{\bar{\mathbf{r}}}$ is the velocity $\bar{\mathbf{v}}$ of the mass center. Therefore, as before, we find that the linear momentum of any mass system, rigid or nonrigid, is

$$\mathbf{G} = m\bar{\mathbf{v}} \qquad [4/5]$$

Next in Art. 4/4 we rewrote Newton's generalized second law as Eq. 4/6. This equation and its integrated form are

$$\Sigma\mathbf{F} = \dot{\mathbf{G}} \quad \text{and} \quad \mathbf{G}_1 + \int_{t_1}^{t_2} \Sigma\mathbf{F}\, dt = \mathbf{G}_2 \qquad \mathbf{(6/12)}$$

Equation 6/12 may be written in its scalar-component form, which, for plane motion in the x-y plane, gives

$$\begin{aligned} \Sigma F_x &= \dot{G}_x \\ \Sigma F_y &= \dot{G}_y \end{aligned} \quad \text{and} \quad \begin{aligned} (G_x)_1 + \int_{t_1}^{t_2} \Sigma F_x\, dt &= (G_x)_2 \\ (G_y)_1 + \int_{t_1}^{t_2} \Sigma F_y\, dt &= (G_y)_2 \end{aligned} \qquad \mathbf{(6/12a)}$$

Angular Momentum

Angular momentum is defined as the moment of linear momentum.

Because the angular-momentum vector is always normal to the plane of motion, vector notation is generally unnecessary, and we may write the angular momentum about the mass center as the scalar

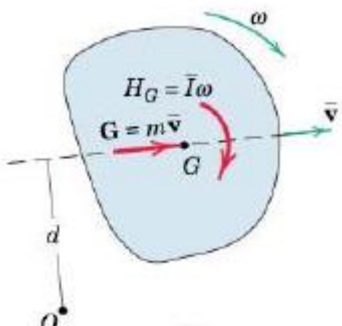
$$H_G = \bar{I}\omega \qquad \textbf{(6/13)}$$

This angular momentum appears in the moment-angular-momentum relation, Eq. 4/9, which in scalar notation for plane motion, along with its integrated form, is

$$\Sigma M_G = \dot{H}_G \quad \text{and} \quad (H_G)_1 + \int_{t_1}^{t_2} \Sigma M_G \, dt = (H_G)_2 \qquad \textbf{(6/14)}$$

This ice skater can effect a large increase in angular speed about a vertical axis by drawing her arms closer to the center of her body.

With the moments about G of the linear momenta of all particles accounted for by $H_G = \bar{I}\omega$, it follows that we may represent the linear momentum $\mathbf{G} = m\bar{\mathbf{v}}$ as a vector through the mass center G, as shown in Fig. 6/14a. Thus, $\mathbf{G}$ and $\mathbf{H}_G$ have vector properties analogous to those of the resultant force and couple.



(a)

With the establishment of the linear- and angular-momentum resultants in Fig. 6/14a, which represents the momentum diagram, the angular momentum H_O about any point O is easily written as

$$H_O = \bar{I}\omega + m\bar{v}d \qquad \textbf{(6/15)}$$

This expression holds at any particular instant of time about O, which may be a fixed or moving point on or off the body.

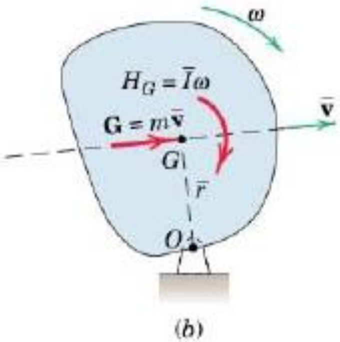


Figure 6/14

When a body rotates about a fixed point O on the body or body extended, as shown in Fig. 6/14b, the relations $v = r\omega$ and $d = r$ may be substituted into the expression for H_O, giving $H_O = (\bar{I}\omega + m\bar{r}^2\omega)$. But $\bar{I} + m\bar{r}^2 = I_O$ so that

$$H_O = I_O\omega \qquad \textbf{(6/16)}$$

In Art. 4/2 we derived Eq. 4/7, which is the moment-angular-momentum equation about a fixed point O. This equation, written in scalar notation for plane motion along with its integrated form, is

$$\Sigma M_O = \dot{H}_O \quad \text{and} \quad (H_O)_1 + \int_{t_1}^{t_2} \Sigma M_O \, dt = (H_O)_2 \qquad \textbf{(6/17)}$$

Note that you should not add linear momentum and angular momentum for the same reason that force and moment cannot be added directly.

Interconnected Rigid Bodies

The equations of impulse and momentum may also be used for a system of interconnected rigid bodies since the momentum principles are applicable to any general system of constant mass. Figure 6/15 shows the combined free-body diagram and momentum diagram for two interconnected bodies a and b. Equations 4/6 and 4/7, which are $\Sigma\mathbf{F} = \dot{\mathbf{G}}$ and $\Sigma\mathbf{M}_O = \dot{\mathbf{H}}_O$ where O is a fixed reference point, may be written for each member of the system and added. The sums are

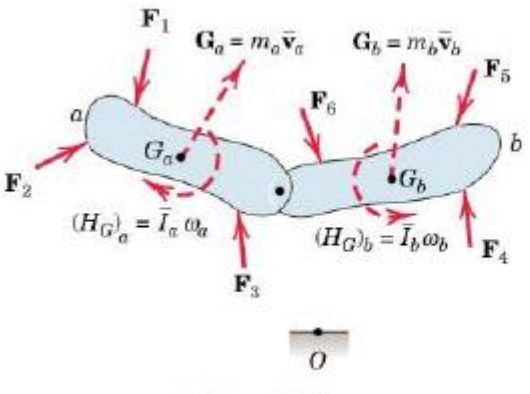


Figure 6/15

$$\begin{aligned} \Sigma\mathbf{F} &= \dot{\mathbf{G}}_a + \dot{\mathbf{G}}_b + \cdots \\ \Sigma\mathbf{M}_O &= (\dot{\mathbf{H}}_O)_a + (\dot{\mathbf{H}}_O)_b + \cdots \end{aligned} \qquad \textbf{(6/18)}$$

In integrated form for a finite time interval, these expressions become

$$\int_{t_1}^{t_2} \Sigma\mathbf{F}\,dt = (\Delta\mathbf{G})_{\text{system}} \qquad \int_{t_1}^{t_2} \Sigma\mathbf{M}_O\,dt = (\Delta\mathbf{H}_O)_{\text{system}} \qquad \textbf{(6/19)}$$

We note that the equal and opposite actions and reactions in the connections are internal to the system and cancel one another so they are not involved in the force and moment summations. Also, point O is one fixed reference point for the entire system.

Conservation of Momentum

In Art. 4/5, we expressed the principles of conservation of momentum for a general mass system by Eqs. 4/15 and 4/16. These principles are applicable to either a single rigid body or a system of interconnected rigid bodies. Thus, if $\Sigma\mathbf{F} = \mathbf{0}$ for a given interval of time, then

$$\mathbf{G}_1 = \mathbf{G}_2 \qquad [4/15]$$

which says that the linear-momentum vector undergoes no change in the absence of a resultant linear impulse. For the system of interconnected rigid bodies, there may be linear-momentum changes of individual parts of the system during the interval, but there will be no resultant momentum change for the system as a whole if there is no resultant linear impulse.

Conservation of Momentum

Similarly, if the resultant moment about a given fixed point O or about the mass center is zero during a particular interval of time for a single rigid body or for a system of interconnected rigid bodies, then

$$(\mathbf{H}_O)_1 = (\mathbf{H}_O)_2 \quad \text{or} \quad (\mathbf{H}_G)_1 = (\mathbf{H}_G)_2 \qquad \textbf{[4/16]}$$

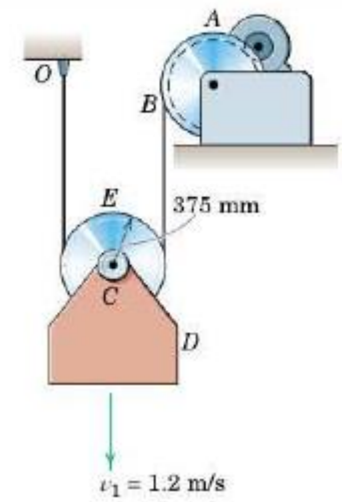
which says that the angular momentum either about the fixed point or about the mass center undergoes no change in the absence of a corresponding resultant angular impulse. Again, in the case of the interconnected system, there may be angular-momentum changes of individual components during the interval, but there will be no resultant angular-momentum change for the system as a whole if there is no resultant angular impulse about the fixed point or the mass center. Either of Eqs. 4/16 may hold without the other.

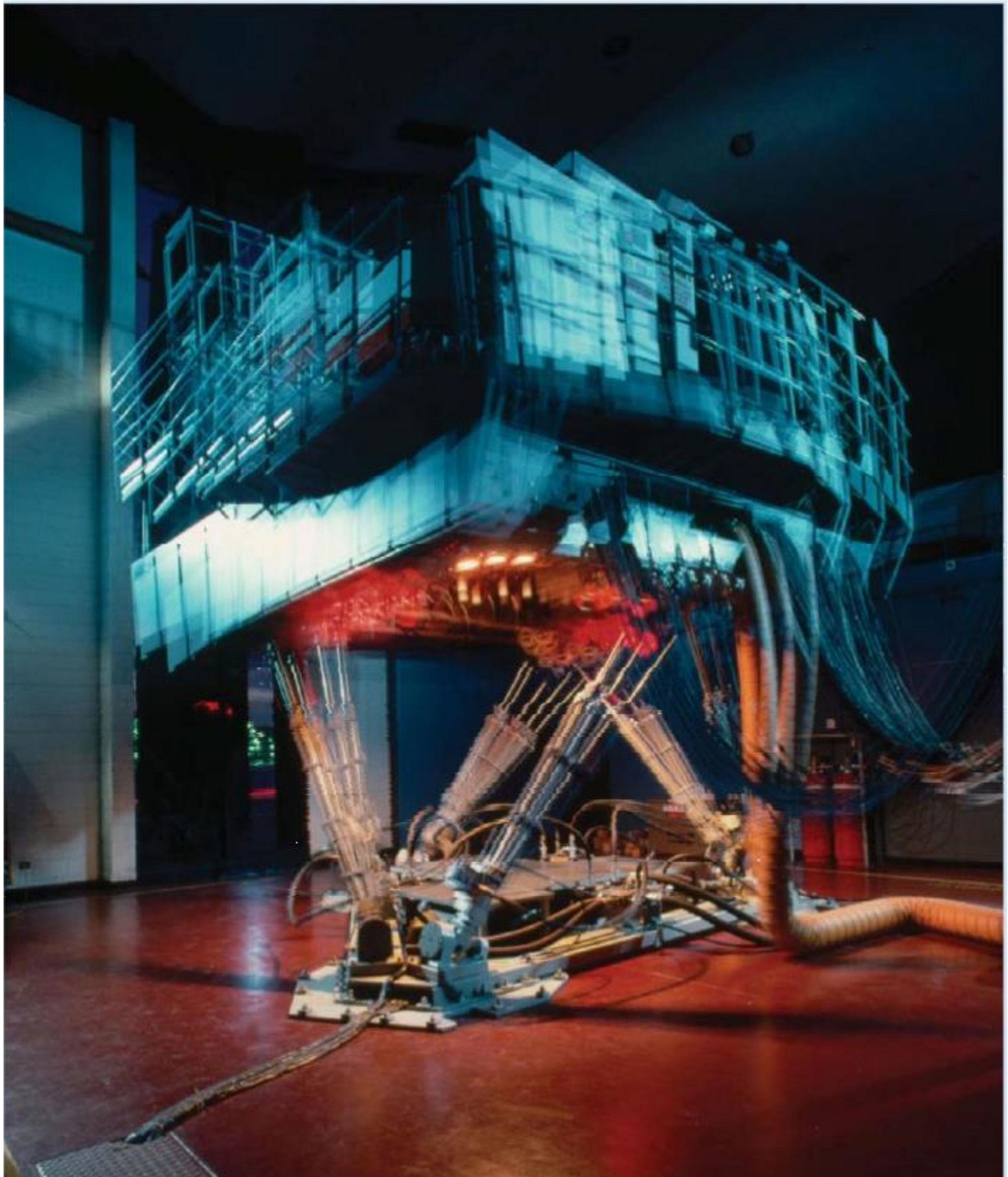
Impact of Rigid Bodies

There are small reaction wheels inside the Hubble Space Telescope that make precision attitude control possible. The principles of angular momentum are fundamental to the design and operation of such a control system.

Sample Problem 6/15

The sheave E of the hoisting rig shown has a mass of 30 kg and a centroidal radius of gyration of 250 mm. The 40-kg load D which is carried by the sheave has an initial downward velocity $v_1 = 1.2$ m/s at the instant when a clockwise torque is applied to the hoisting drum A to maintain essentially a constant force $F = 380$ N in the cable at B. Compute the angular velocity ω_2 of the sheave 5 seconds after the torque is applied to the drum and find the tension T in the cable at O during the interval. Neglect all friction.





By proper management of the hydraulic cylinders which support and move this flight simulator, a variety of three-dimensional translational and rotational accelerations can be produced.

References

- J.L. Meriam and L. G. Krage, Dynamics 6^{TH} edition