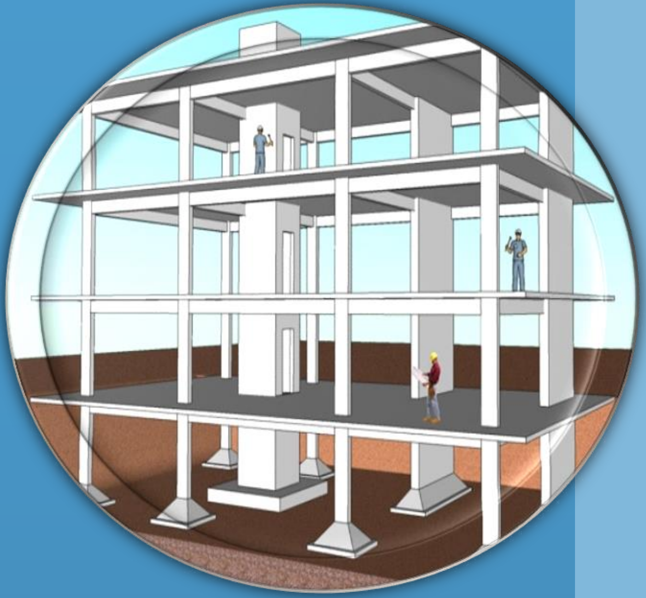
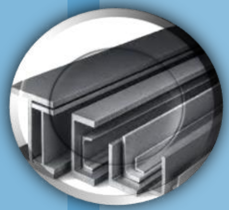
UNIVERSITY OF ZAMBIA
School of Engineering
Department of Civil & Environmental Eng.





Lecture 3.1

CEE 3222: THEORY OF STRUCTURES

ENERGY METHODS-DELECTION

Contents

Introduction

- In this chapter we will show how to determine the elastic deflections of a beam using the method of double integration and two important geometrical methods, namely, the moment-area theorems and the conjugate-beam method. Double integration is used to obtain equations which define the slope and the elastic curve. The geometric methods provide a way to obtain the slope and deflection at specific points on the beam. Each of these methods has particular advantages or disadvantages, which will be discussed when each method is presented.

Deflection Curves

Deflection curves

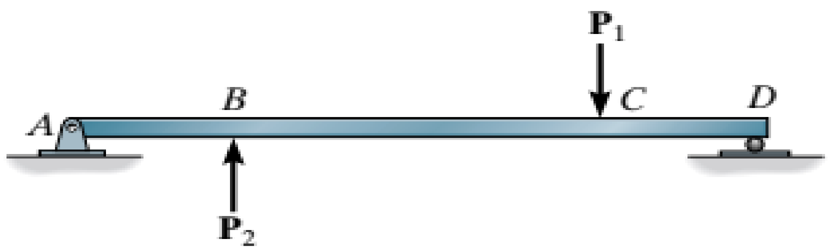
- Deflections of structures can occur from various sources, such as loads, temperature, fabrication errors, or settlement.
- Deflections must be limited in order to provide integrity and stability of roofs, and prevent cracking of attached brittle materials such as concrete, plaster or glass.
- A structure must not vibrate or deflect severely in order to "appear" safe for its occupants.
- Deflections at specified points in a structure must be determined to analyze statically indeterminate structures.

Deflection curves

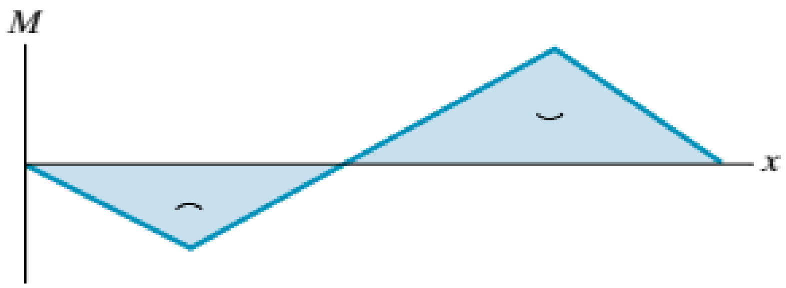
- We consider having linear elastic material response.
- For beams and frames, deflections are most often caused by internal bending
- Internal axial forces cause the deflections of a truss.
- The deflection diagram represents the elastic curve or locus of points which defines the displaced position of the centroid of the cross section along the members.
- Supports that resist a force, such as a pin, restrict displacement; and those that resist moment, such as a fixed wall, restrict rotation.

Deflection curves

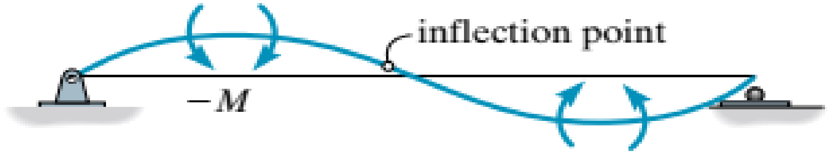
- If the shape of the moment diagram is known, it will be easy to construct the elastic curve and vice versa.
- In this particular beam, there must be an **inflection point** at the point where the curve changes from concave down to concave up, since this is a point of zero moment.



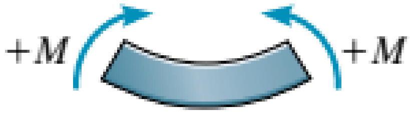
beam



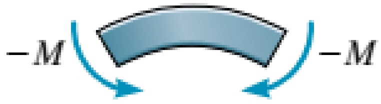
moment diagram



deflection curve



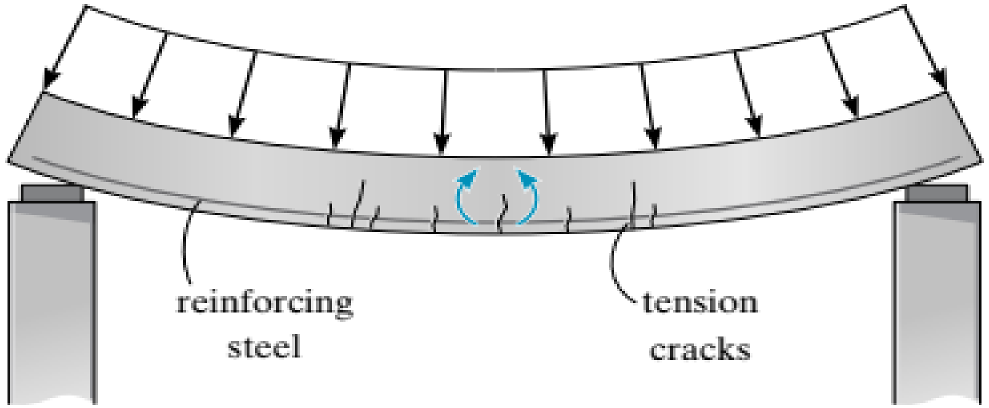
positive moment,
concave upward



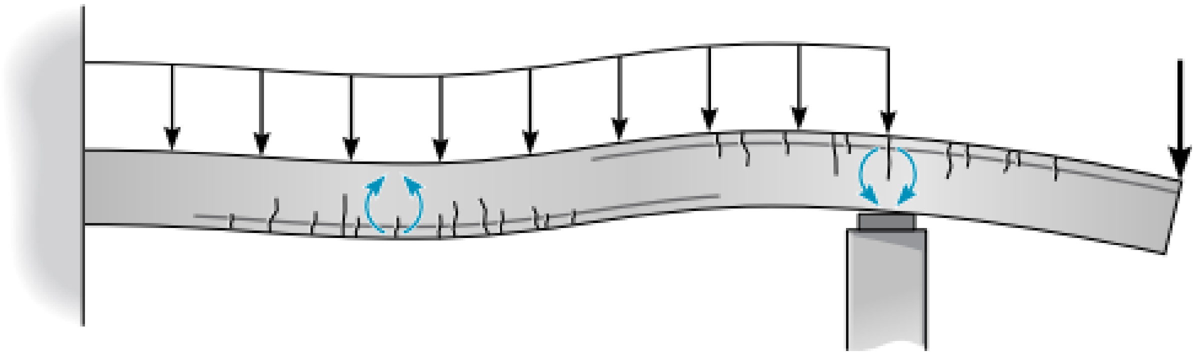
negative moment,
concave downward

Deflection curves

- The deflection curve also helps engineers in locating the steel needed to reinforce a concrete beam, column, or wall.
- In concrete, they are used to determine the location and quantity of reinforcing rods



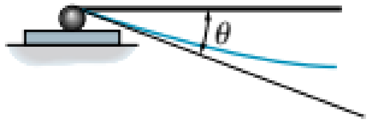
simply supported beam



overhang beam

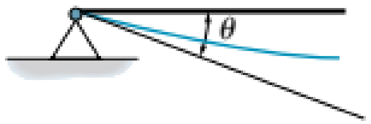
Deflection curves

(1)



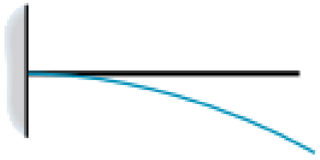
$\Delta = 0$
roller or rocker

(2)

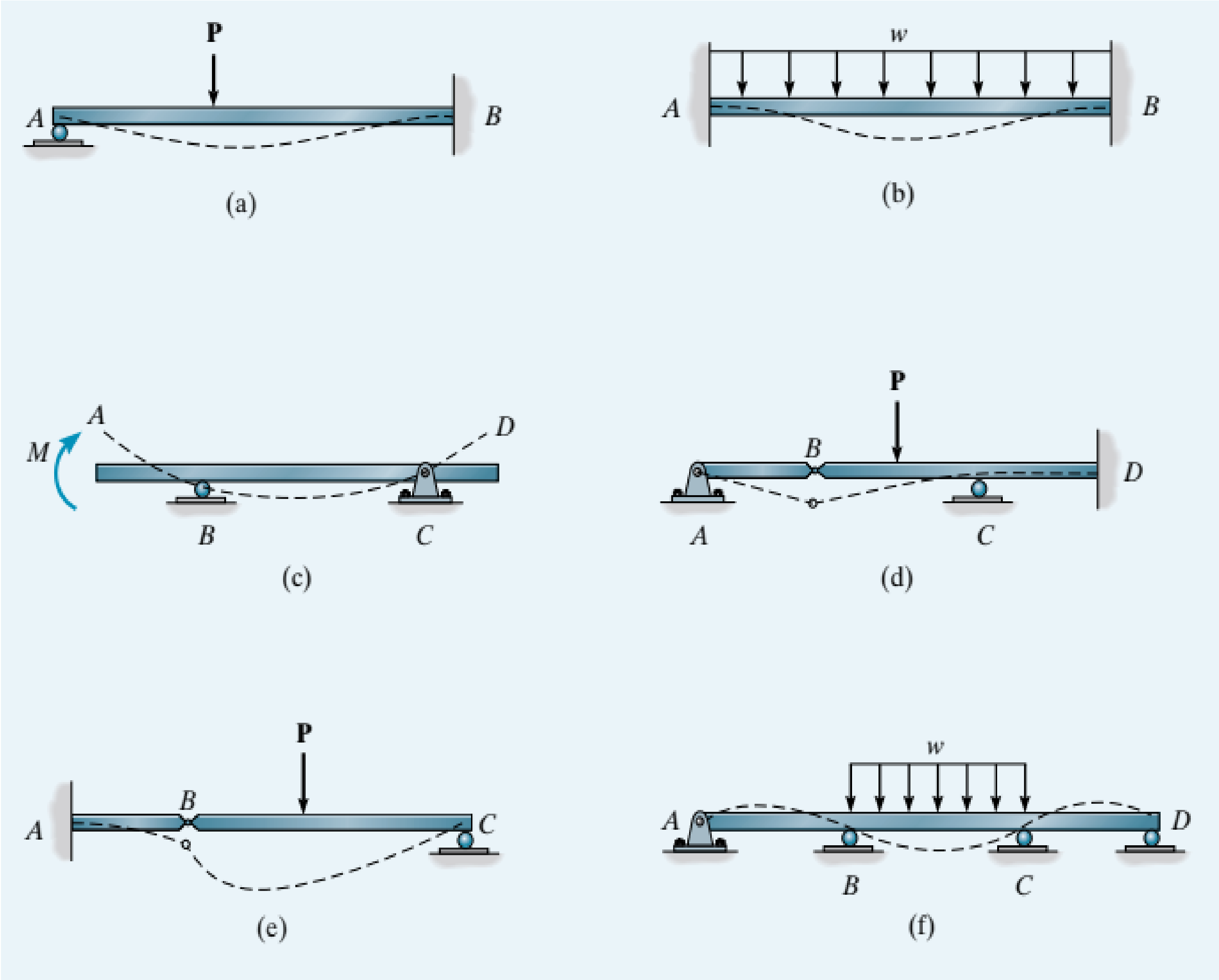


$\Delta = 0$
pin

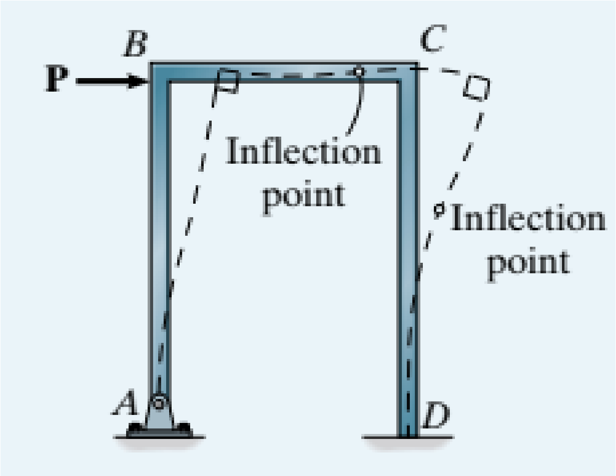
(3)

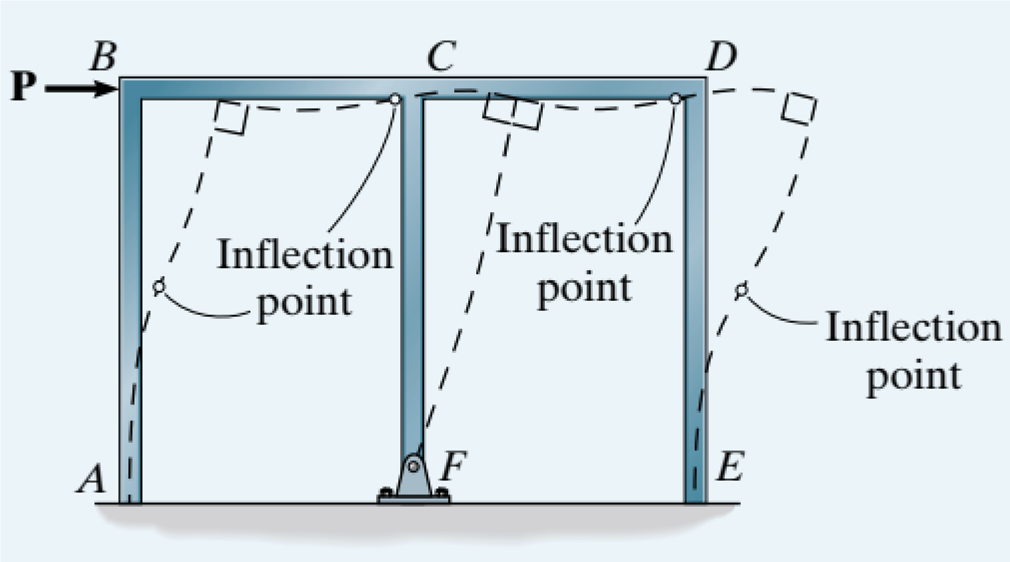


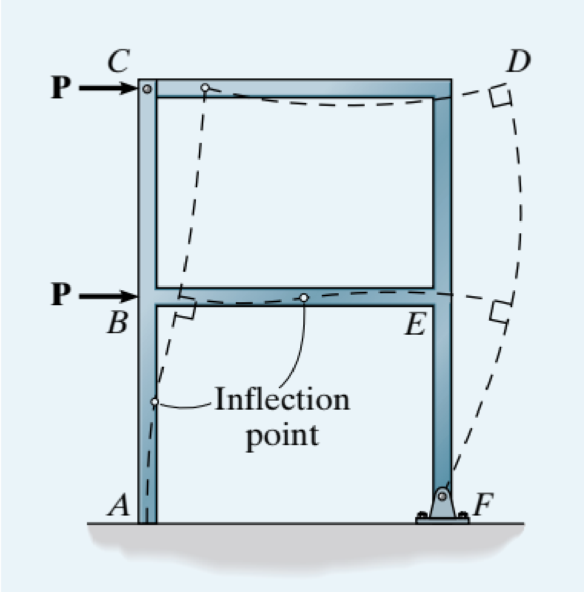
$\Delta = 0$
$\theta = 0$
fixed support

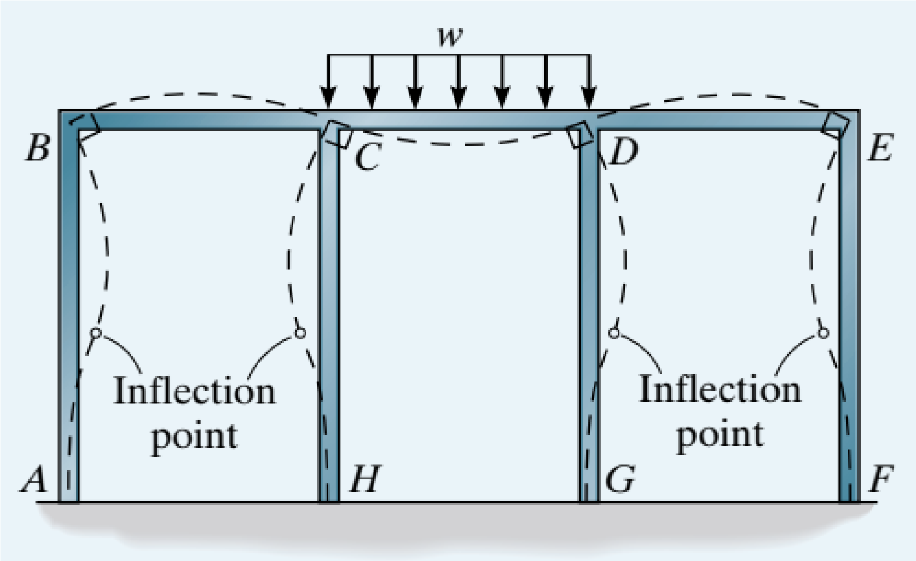


Deflection curves









Elastic Beam Theory

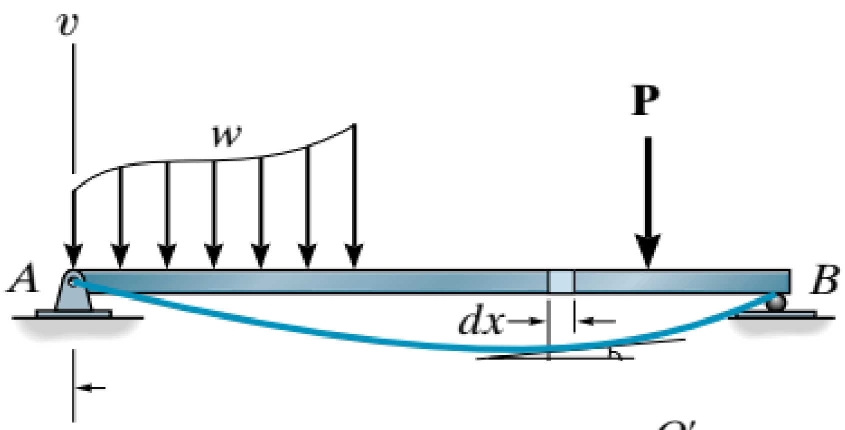
Elastic-Beam Theory

- We create the relationship between the internal moment and the displacement and slope of its elastic curve.
- From the figure, the strain at the neutral axis is given by

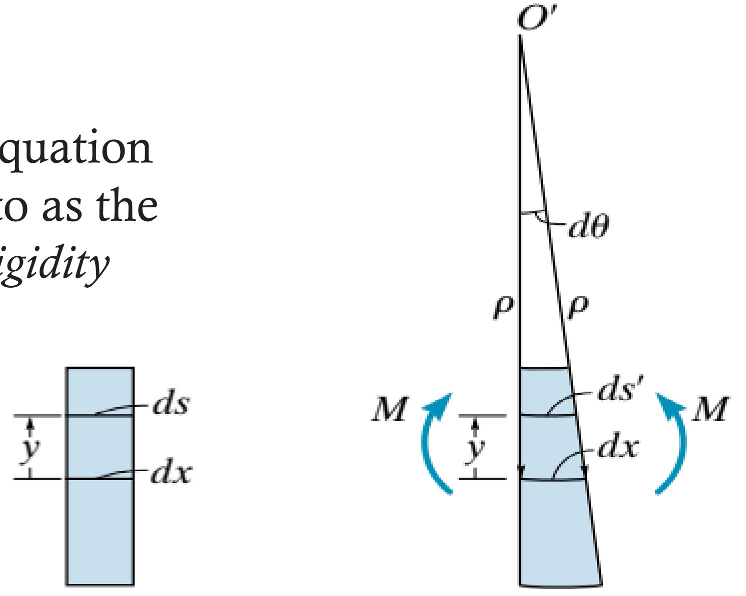
$$\epsilon = \frac{ds' - ds}{ds} = \frac{(\rho - y)d\theta - \rho d\theta}{\rho d\theta} = -\frac{y}{\rho}$$

- But $\epsilon = \sigma / E$ and $\sigma = -My/I$

$$\frac{1}{\rho} = \frac{M}{EI} \text{ and thus } d\theta = 1/\rho d\theta = \frac{M}{EI}$$



EI in this equation is referred to as the *flexural rigidity*



before deformation

after deformation

$$\frac{1}{\rho} = \frac{d^2v/dx^2}{\left[1 + (dv/dx)^2\right]^{3/2}}$$

$$\frac{M}{EI} = \frac{d^2v/dx^2}{\left[1 + (dv/dx)^2\right]^{3/2}}$$

Curvature

assume $dv/dx \approx 0$

$$ds = \sqrt{dx^2 + dv^2} = \sqrt{1 + (dv/dx)^2}\, dx \approx dx$$

$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$

Double Integration Method

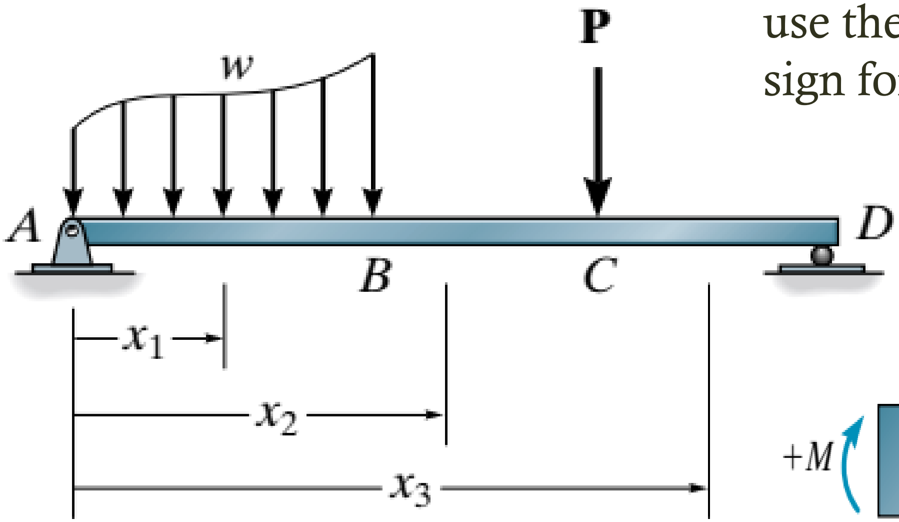
Double Integration Method

- M is expressed as a function of position x, then successive integrations yields

$$\theta = \tan\theta = \frac{dv}{dx} = \int \frac{M}{EI}\,dx$$

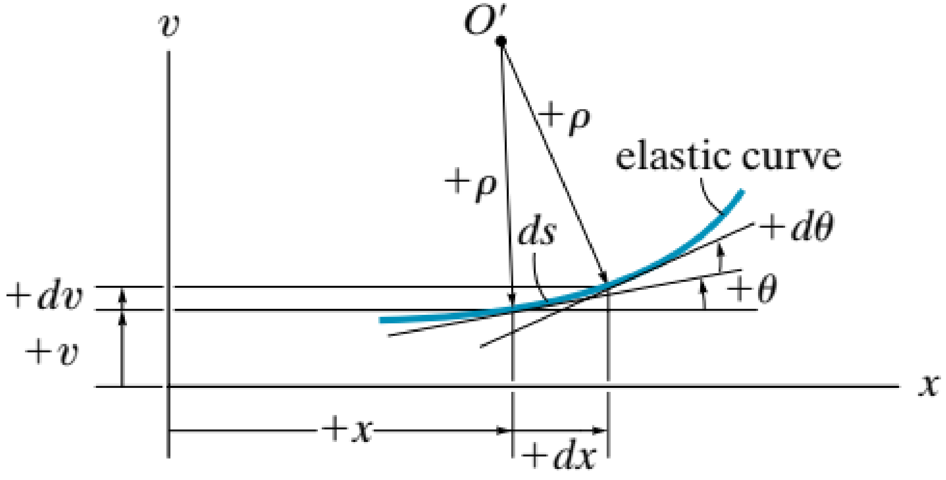
Thus

$$v = \int\int \frac{M}{EI}\,dx\,dx$$



- it is important to use the proper sign for M

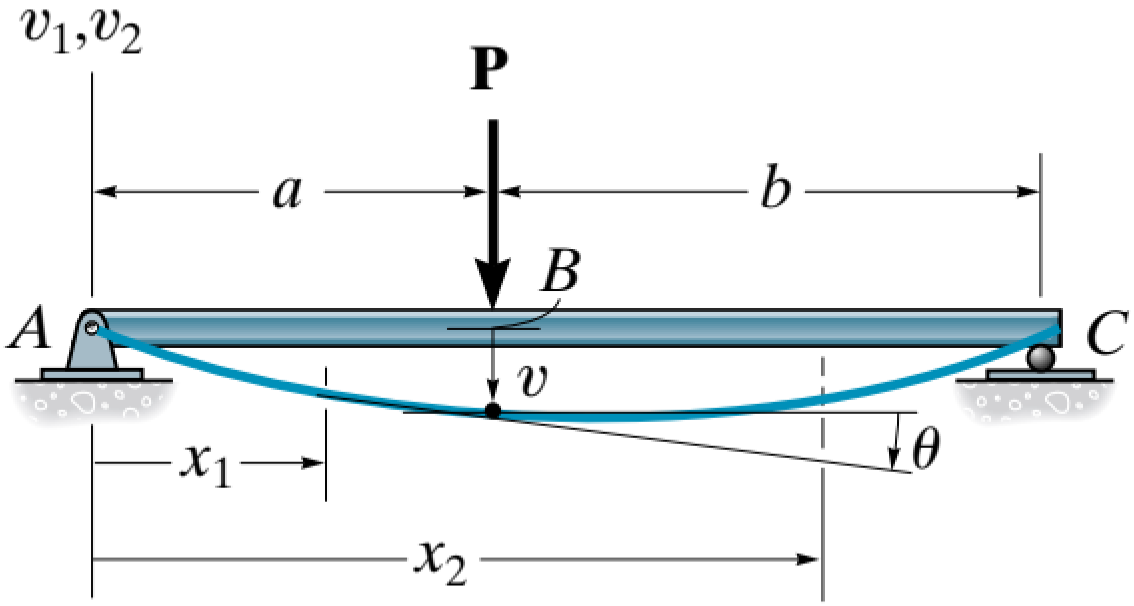
- positive deflection, v is upward, and as a result, the positive slope angle $\boldsymbol{\theta}$ will be measured counterclockwise from the x axis.
- If the beam is supported by a roller or pin, then it is required that the displacement be zero at these points. Also, at a fixed support the slope and displacement are both zero.



Double Integration Method

- If the beam is supported by a roller or pin, then it is required that the displacement be zero at these points. Also, at a fixed support the slope and displacement are both zero.

- If a single x coordinate cannot be used to express the equation for the beam's slope or the elastic curve, then continuity conditions must be used to evaluate some of the integration constants.



Double Integration Method

Procedure for Analysis

The following procedure provides a method for determining the slope and deflection of a beam (or shaft) using the method of double integration. It should be realized that this method is suitable only for *elastic deflections* for which the beam's slope is very small. Furthermore, the method considers *only deflections due to bending*. Additional deflection due to shear generally represents only a few percent of the bending deflection, and so it is usually neglected in engineering practice.

Elastic Curve

- Draw an exaggerated view of the beam's elastic curve. Recall that points of zero slope and zero displacement occur at a fixed support, and zero displacement occurs at pin and roller supports.
- Establish the x and v coordinate axes. The x axis must be parallel to the undeflected beam and its origin at the left side of the beam, with a positive direction to the right.
- If several discontinuous loads are present, establish x coordinates that are valid for each region of the beam between the discontinuities.
- In all cases, the associated positive v axis should be directed upward.

Load or Moment Function

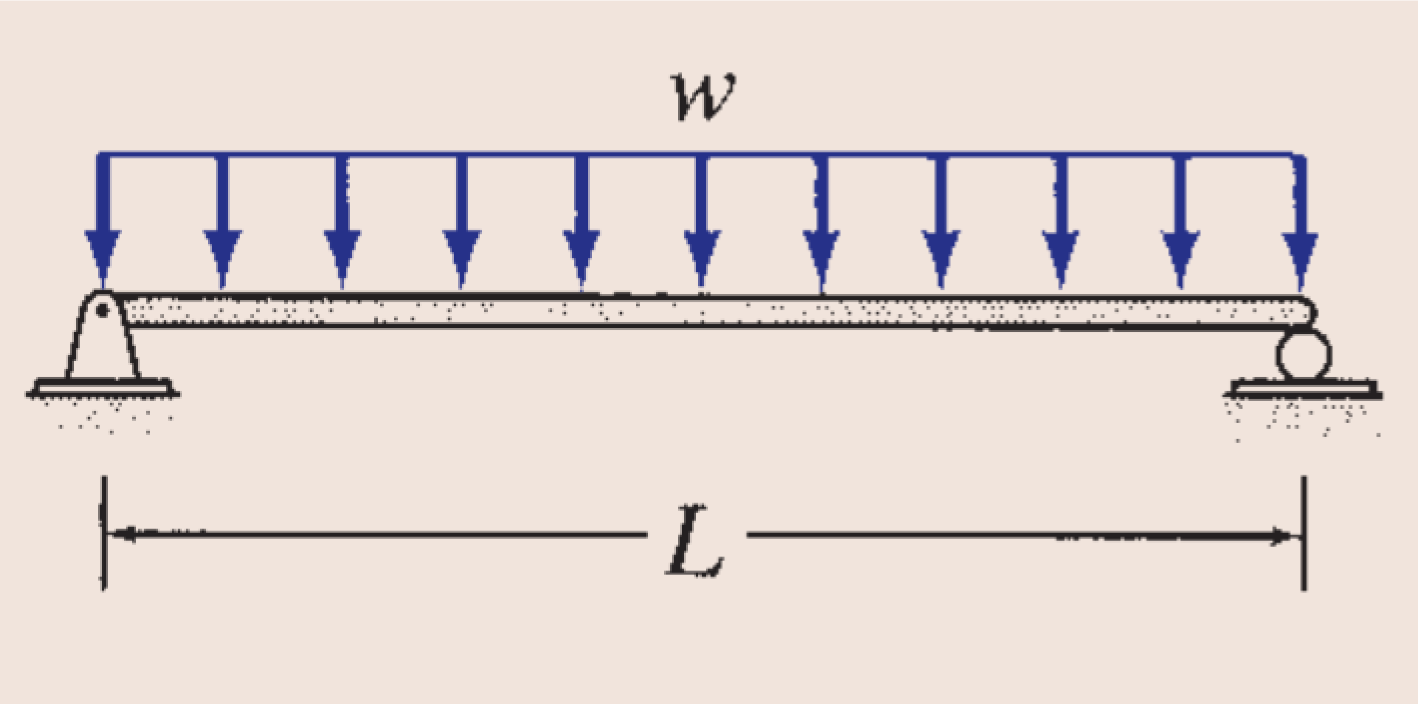
- For each region in which there is an x coordinate, express the internal moment M as a function of x.
- *Always* assume that M acts in the positive direction when applying the equation of moment equilibrium to determine $M = f(x)$.

Slope and Elastic Curve

- Provided EI is constant, apply the moment equation $EI\,d^2v/dx^2 = M(x)$, which requires two integrations. For each integration it is important to include a constant of integration. The constants are determined using the boundary conditions for the supports and the continuity conditions that apply to slope and displacement at points where two functions meet.
- Once the integration constants are determined and substituted back into the slope and deflection equations, the slope and displacement at *specific points* on the elastic curve can be determined. The numerical values obtained can be checked graphically by comparing them with the sketch of the elastic curve.
- *Positive* values for *slope* are *counterclockwise* and *positive displacement is upward.*

Example 3.1

- Determine the equations for the slope and deflection of the beam shown in the figure below by the direct integration method. Also, compute the slope at each end and the deflection at the midspan of the beam. EI is constant.



Solution 3.1

Reactions. See Fig. 6.2(b).

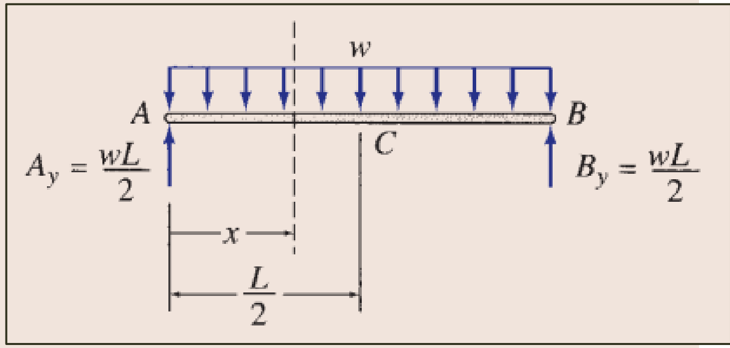
$$+\rightarrow \sum F_x = 0 \qquad A_x = 0$$

$$+\circlearrowleft \sum M_B = 0$$

$$-A_y(L) + w(L)\left(\frac{L}{2}\right) = 0 \qquad A_y = \frac{wL}{2}\uparrow$$

$$+\uparrow \sum F_y = 0$$

$$\left(\frac{wL}{2}\right) - (wL) + B_y = 0 \qquad B_y = \frac{wL}{2}\uparrow$$



Equation for Bending Moment. To determine the equation for bending moment for the beam, we pass a section at a distance x from support A, as shown in Fig. 6.2(b). Considering the free body to the left of this section, we obtain

$$M = \frac{wL}{2}(x) - (wx)\left(\frac{x}{2}\right) = \frac{w}{2}(Lx - x^2)$$

Equation for M/EI. The flexural rigidity, EI, of the beam is constant, so the equation for M/EI can be written as

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{w}{2EI}(Lx - x^2)$$

Solution 3.1 cont.

Equations for Slope and Deflection. The equation for the slope of the elastic curve of the beam can be obtained by integrating the equation for M/EI as

$$\theta = \frac{dy}{dx} = \frac{w}{2EI}\left(\frac{Lx^2}{2} - \frac{x^3}{3}\right) + C_1$$

Integrating once more, we obtain the equation for deflection as

$$y = \frac{w}{2EI}\left(\frac{Lx^3}{6} - \frac{x^4}{12}\right) + C_1x + C_2$$

The constants of integration, C_1 and C_2, are evaluated by applying the following boundary conditions:

$$\begin{array}{lll} \text{At end } A, & x = 0, & y = 0 \\ \text{At end } B, & x = L, & y = 0 \end{array}$$

By applying the first boundary condition—that is, by setting $x = 0$ and $y = 0$ in the equation for y—we obtain $C_2 = 0$. Next, by using the second boundary condition—that is, by setting $x = L$ and $y = 0$ in the equation for y—we obtain

$$0 = \frac{w}{2EI}\left(\frac{L^4}{6} - \frac{L^4}{12}\right) + C_1L$$

from which

$$C_1 = -\frac{wL^3}{24EI}$$

Solution 3.1 cont.

Thus, the equations for slope and deflection of the beam are

$$\theta = \frac{w}{2EI}\left(\frac{Lx^2}{2} - \frac{x^3}{3} - \frac{L^3}{12}\right) \qquad (1) \quad \text{Ans.}$$

$$y = \frac{wx}{12EI}\left(Lx^2 - \frac{x^3}{2} - \frac{L^3}{2}\right) \qquad (2) \quad \text{Ans.}$$

Slopes at Ends *A* and *B*. By substituting $x = 0$ and L, respectively, into Eq. (1), we obtain

$$\theta_A = -\frac{wL^3}{24EI} \quad \text{or} \quad \theta_A = \frac{wL^3}{24EI} \qquad \text{Ans.}$$

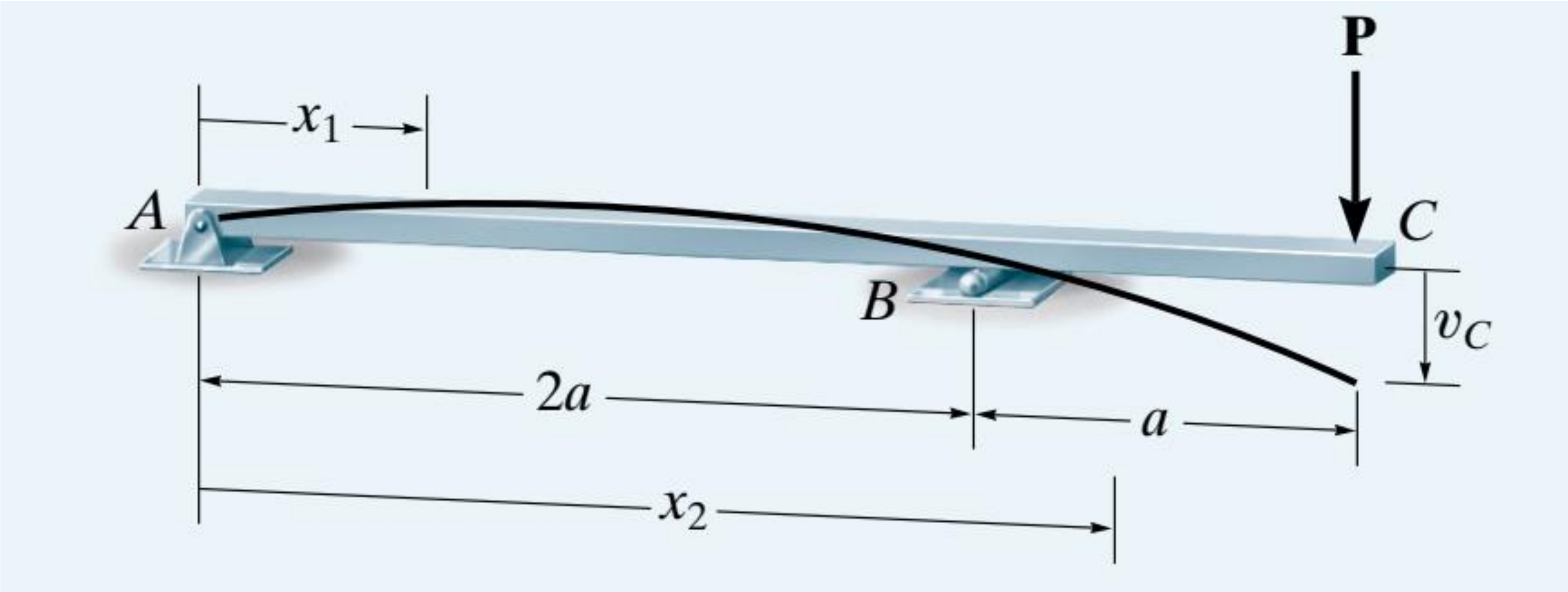
$$\theta_B = \frac{wL^3}{24EI} \quad \text{or} \quad \theta_B = \frac{wL^3}{24EI} \qquad \text{Ans.}$$

Deflection at Midspan. By substituting $x = L/2$ into Eq. (2), we obtain

$$y_C = -\frac{5wL^4}{384EI} \quad \text{or} \quad y_C = \frac{5wL^4}{384EI} \downarrow \qquad \text{Ans.}$$

Example 3.2

- The beam in the figure is subjected to a load P at its end. Determine the displacement at C. EI is constant.



Solution 3.2

Elastic Curve. The beam deflects into the shape shown in Fig. 8–14*a*. Due to the loading, two x coordinates must be considered.

Moment Functions. Using the free-body diagrams shown in Fig. 8–14*b*, we have

$$M_1 = -\frac{P}{2}x_1 \qquad 0 \le x_1 \le 2a$$

$$M_2 = -\frac{P}{2}x_2 + \frac{3P}{2}(x_2 - 2a)$$

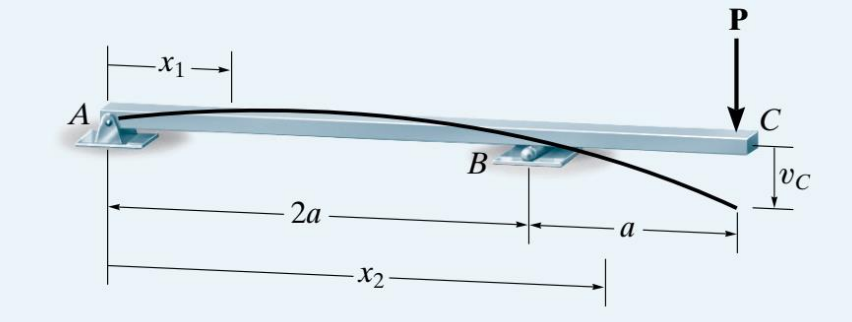
$$= Px_2 - 3Pa \qquad 2a \le x_2 \le 3a$$

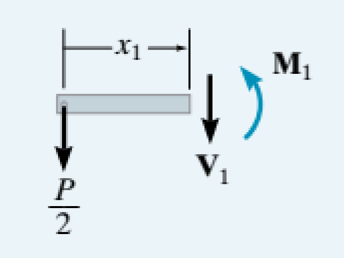
Slope and Elastic Curve. Applying Eq. 8–4,

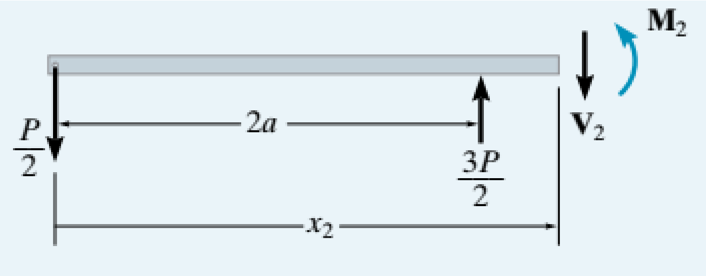
for x_1,
$$EI\frac{d^2v_1}{dx_1^2} = -\frac{P}{2}x_1$$

$$EI\frac{dv_1}{dx_1} = -\frac{P}{4}x_1^2 + C_1 \qquad (1)$$

$$EIv_1 = -\frac{P}{12}x_1^3 + C_1x_1 + C_2 \qquad (2)$$







Solution 3.2

For x_2, $$EI\frac{d^2v_2}{dx_2^2} = Px_2 - 3Pa$$

$$EI\frac{dv_2}{dx_2} = \frac{P}{2}x_2^2 - 3Pax_2 + C_3 \tag{3}$$

$$EIv_2 = \frac{P}{6}x_2^3 - \frac{3}{2}Pax_2^2 + C_3x_2 + C_4 \tag{4}$$

The *four* constants of integration are determined using *three* boundary conditions, namely, $v_1 = 0$ at $x_1 = 0$, $v_1 = 0$ at $x_1 = 2a$, and $v_2 = 0$ at $x_2 = 2a$, and *one* continuity equation. Here the continuity of slope at the roller requires $dv_1/dx_1 = dv_2/dx_2$ at $x_1 = x_2 = 2a$. (Note that continuity of displacement at B has been indirectly considered in the boundary conditions, since $v_1 = v_2 = 0$ at $x_1 = x_2 = 2a$.) Applying these four conditions yields

$v_1 = 0$ at $x_1 = 0$; $\quad 0 = 0 + 0 + C_2$

$v_1 = 0$ at $x_1 = 2a$; $\quad 0 = -\frac{P}{12}(2a)^3 + C_1(2a) + C_2$

$v_2 = 0$ at $x_2 = 2a$; $\quad 0 = \frac{P}{6}(2a)^3 - \frac{3}{2}Pa(2a)^2 + C_3(2a) + C_4$

$$\frac{dv_1(2a)}{dx_1} = \frac{dv_2(2a)}{dx_2}; \qquad -\frac{P}{4}(2a)^2 + C_1 = \frac{P}{2}(2a)^2 - 3Pa(2a) + C_3$$

Solving, we obtain

$$C_1 = \frac{Pa^2}{3} \qquad C_2 = 0 \qquad C_3 = \frac{10}{3}Pa^2 \qquad C_4 = -2Pa^3$$

Substituting C_3 and C_4 into Eq. (4) gives

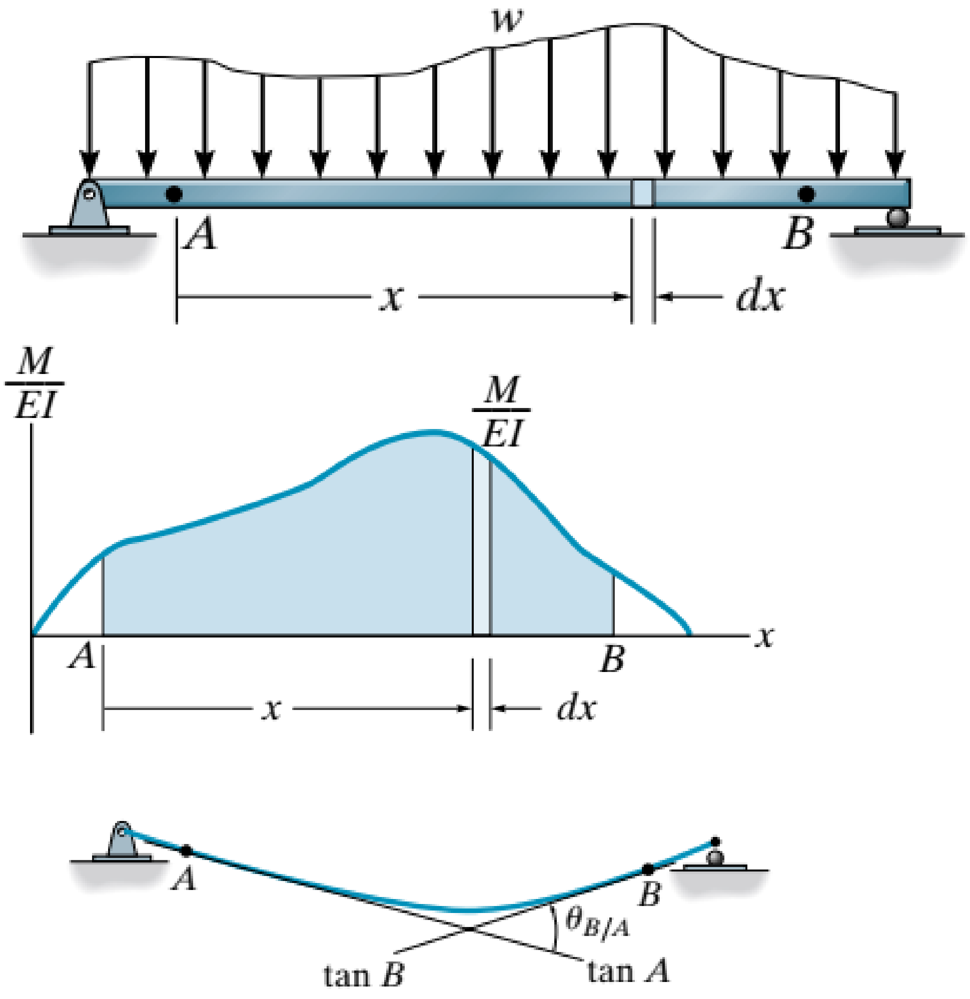
$$v_2 = \frac{P}{6EI}x_2^3 - \frac{3Pa}{2EI}x_2^2 + \frac{10Pa^2}{3EI}x_2 - \frac{2Pa^3}{EI}$$

The displacement at C is determined by setting $x_2 = 3a$. We get

$$v_C = -\frac{Pa^3}{EI} \qquad \textit{Ans.}$$

Moment-Area Theorems

Moment-Area Theorems



- Theorem 1: The change in slope between any two points on the elastic curve equals the area of the M/EI diagram between these two points.

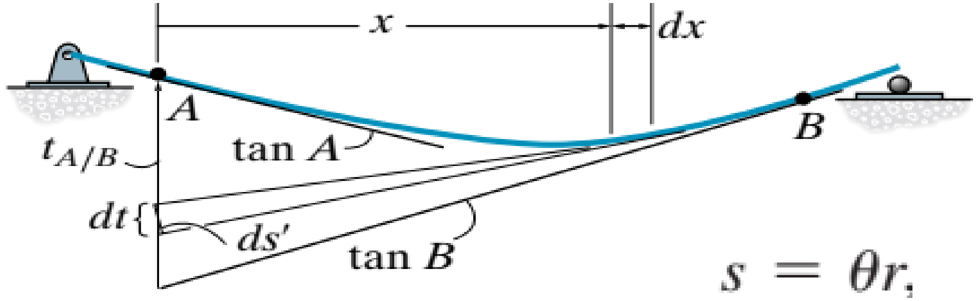
$$d\theta = \left(\frac{M}{EI}\right) dx$$

$$\theta_{B/A} = \int_A^B \frac{M}{EI} dx$$

The notation $\theta_{B/A}$ is referred to as the angle of the tangent at B measured with respect to the tangent at A. *If* $\theta_{B/A}$ is counterclockwise, M/EI diagram is +ve and vice versa. $\theta_{B/A}$ is measured in radians

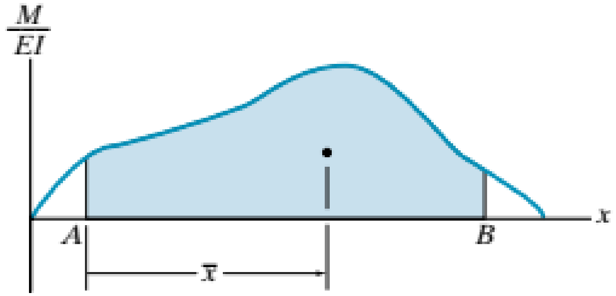
Moment-Area Theorems

length of each tangent line by x and the arc ds' by dt.



$$s = \theta r,$$

$$dt = x\, d\theta \quad d\theta = (M/EI)\, dx$$

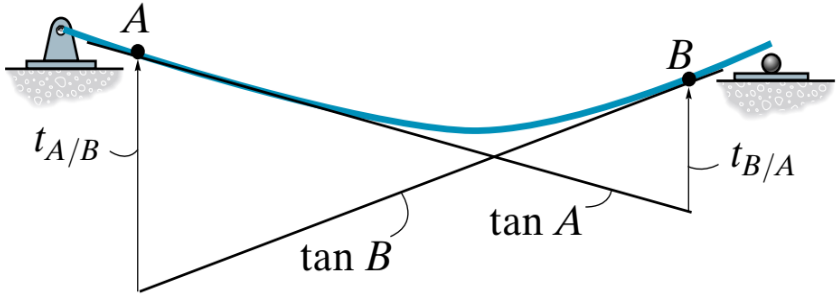


$$t_{A/B} = \int_A^B x \frac{M}{EI}\, dx$$

- Theorem 2: The vertical deviation of the tangent at a point (A) on the elastic curve with respect to the tangent extended from another point (B) equals the "moment" of the area under the M/EI diagram between the two points (A and B). This moment is computed about point A (the point on the elastic curve), where the deviation $t_{A/B}$ is to be determined.

- Recall from statics that the centroid of an area is determined from $\bar{x}\int dA = \int x dA$. Since $\int \frac{M}{EI} dx$ is an area of the M/EI diagram, we can also write

$$t_{A/B} = \bar{x} \int_A^B \frac{M}{EI}\, dx$$



Moment-Area Theorems

Procedure for Analysis

The following procedure provides a method that may be used to determine the displacement and slope at a point on the elastic curve of a beam using the moment-area theorems.

M/EI Diagram

- Determine the support reactions and draw the beam's M/EI diagram.
- If the beam is loaded with concentrated forces, the M/EI diagram will consist of a series of straight line segments, and the areas and their moments required for the moment-area theorems will be relatively easy to compute.
- If the loading consists of a series of concentrated forces and distributed loads, it may be simpler to compute the required M/EI areas and their moments by drawing the M/EI diagram in parts, using the method of superposition as discussed in Sec. 4.5. In any case, the M/EI diagram will consist of parabolic or perhaps higher-order curves, and it is suggested that the table on the inside back cover be used to locate the area and centroid under each curve.

Elastic Curve

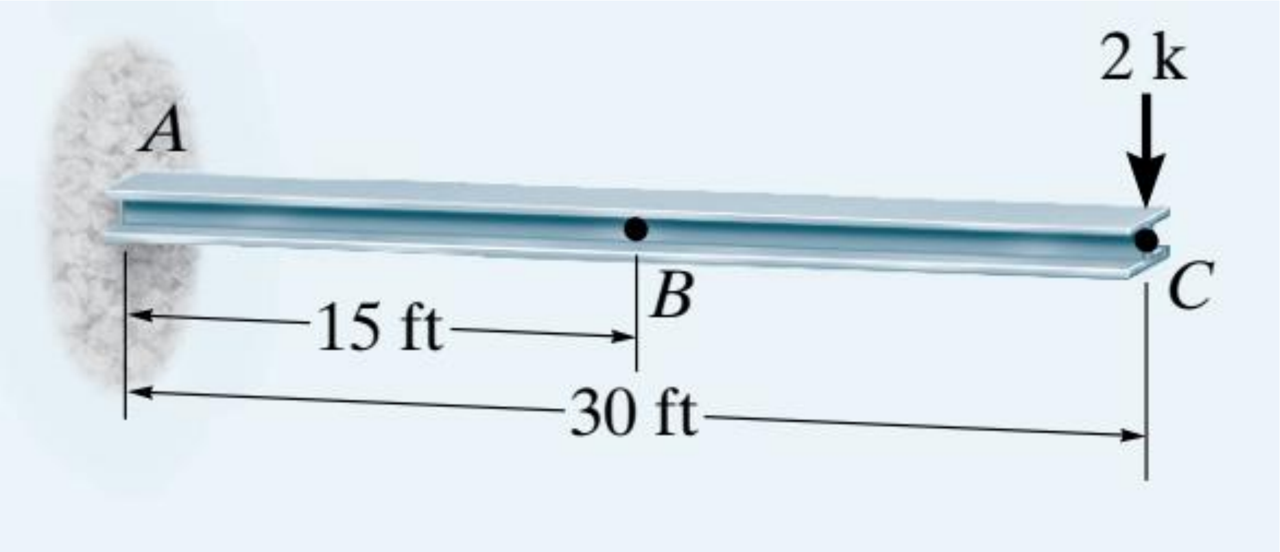
- Draw an exaggerated view of the beam's elastic curve. Recall that points of zero slope occur at fixed supports and zero displacement occurs at all fixed, pin, and roller supports.
- If it becomes difficult to draw the general shape of the elastic curve, use the moment (or M/EI) diagram. Realize that when the beam is subjected to a *positive moment* the beam bends *concave up*, whereas *negative moment* bends the beam *concave down*. Furthermore, an inflection point or change in curvature occurs where the moment in the beam (or M/EI) is zero.
- The displacement and slope to be determined should be indicated on the curve. Since the moment-area theorems apply only between two tangents, attention should be given as to which tangents should be constructed so that the angles or deviations between them will lead to the solution of the problem. In this regard, *the tangents at the points of unknown slope and displacement and at the supports should be considered*, since the beam usually has zero displacement and/or zero slope at the supports.

Moment-Area Theorems

- Apply Theorem 1 to determine the angle between two tangents, and Theorem 2 to determine vertical deviations between these tangents.
- Realize that Theorem 2 in general *will not* yield the displacement of a point on the elastic curve. When applied properly, it will only give the vertical distance or deviation of a tangent at point A on the elastic curve from the tangent at B.
- After applying either Theorem 1 or Theorem 2, the algebraic sign of the answer can be verified from the angle or deviation as indicated on the elastic curve.

Example 3.3

Determine the slope at points B and C of the beam shown. Take $E = 29(10^3)$ ksi and $I = 600\text{ in}^4$.



Solution 3.3

M/EI Diagram. This diagram is shown in Fig. 8–16*b*. It is easier to solve the problem in terms of EI and substitute the numerical data as a last step.

Elastic Curve. The 2-k load causes the beam to deflect as shown in Fig. 8–16*c*. (The beam is deflected concave down, since M/EI is negative.) Here the tangent at A (the support) is *always horizontal*. The tangents at B and C are also indicated. We are required to find θ_B and θ_C. By the construction, the angle between tan A and tan B, that is, $\theta_{B/A}$, is equivalent to θ_B.

$$\theta_B = \theta_{B/A}$$

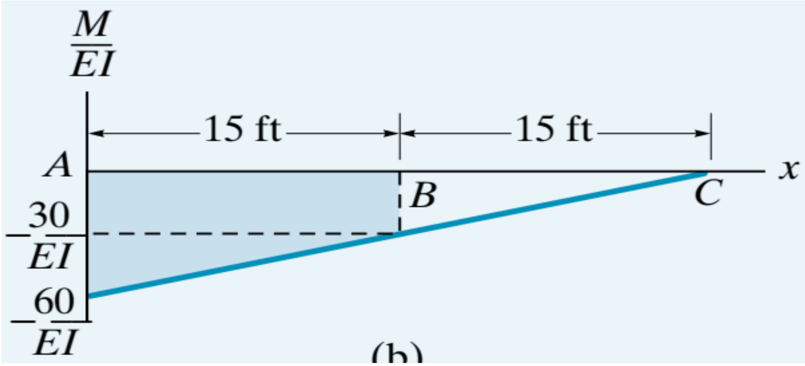
Also,

$$\theta_C = \theta_{C/A}$$

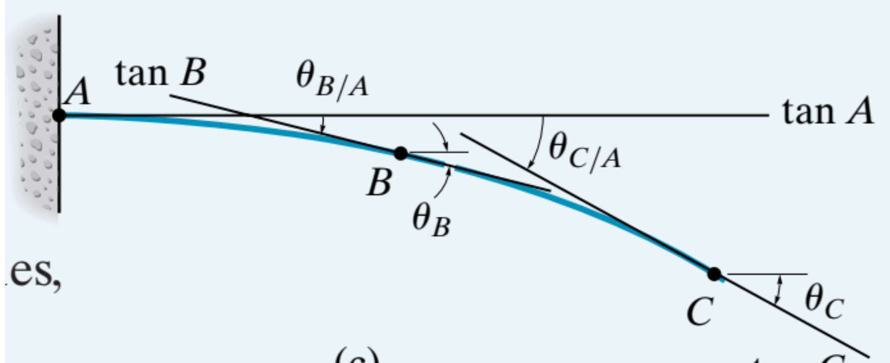
Moment-Area Theorem. Applying Theorem 1, $\theta_{B/A}$ is equal to the area under the M/EI diagram between points A and B; that is,

$$\theta_B = \theta_{B/A} = -\left(\frac{30\ \text{k}\cdot\text{ft}}{EI}\right)(15\ \text{ft}) - \frac{1}{2}\left(\frac{60\ \text{k}\cdot\text{ft}}{EI} - \frac{30\ \text{k}\cdot\text{ft}}{EI}\right)(15\ \text{ft})$$

$$= -\frac{675\ \text{k}\cdot\text{ft}^2}{EI}$$



(b)



(c)

Solution 3.3

Substituting numerical data for E and I, and converting feet to inches, we have

$$\theta_B = \frac{-675\ \text{k}\cdot\text{ft}^2(144\ \text{in}^2/1\ \text{ft}^2)}{29(10^3)\ \text{k/in}^2(600\ \text{in}^4)}$$

$$= -0.00559\ \text{rad} \qquad \textit{Ans.}$$

The *negative sign* indicates that the angle is measured clockwise from A, Fig. 8–16*c*.

In a similar manner, the area under the M/EI diagram between points A and C equals $\theta_{C/A}$. We have

$$\theta_C = \theta_{C/A} = \frac{1}{2}\left(-\frac{60\ \text{k}\cdot\text{ft}}{EI}\right)(30\ \text{ft}) = -\frac{900\ \text{k}\cdot\text{ft}^2}{EI}$$

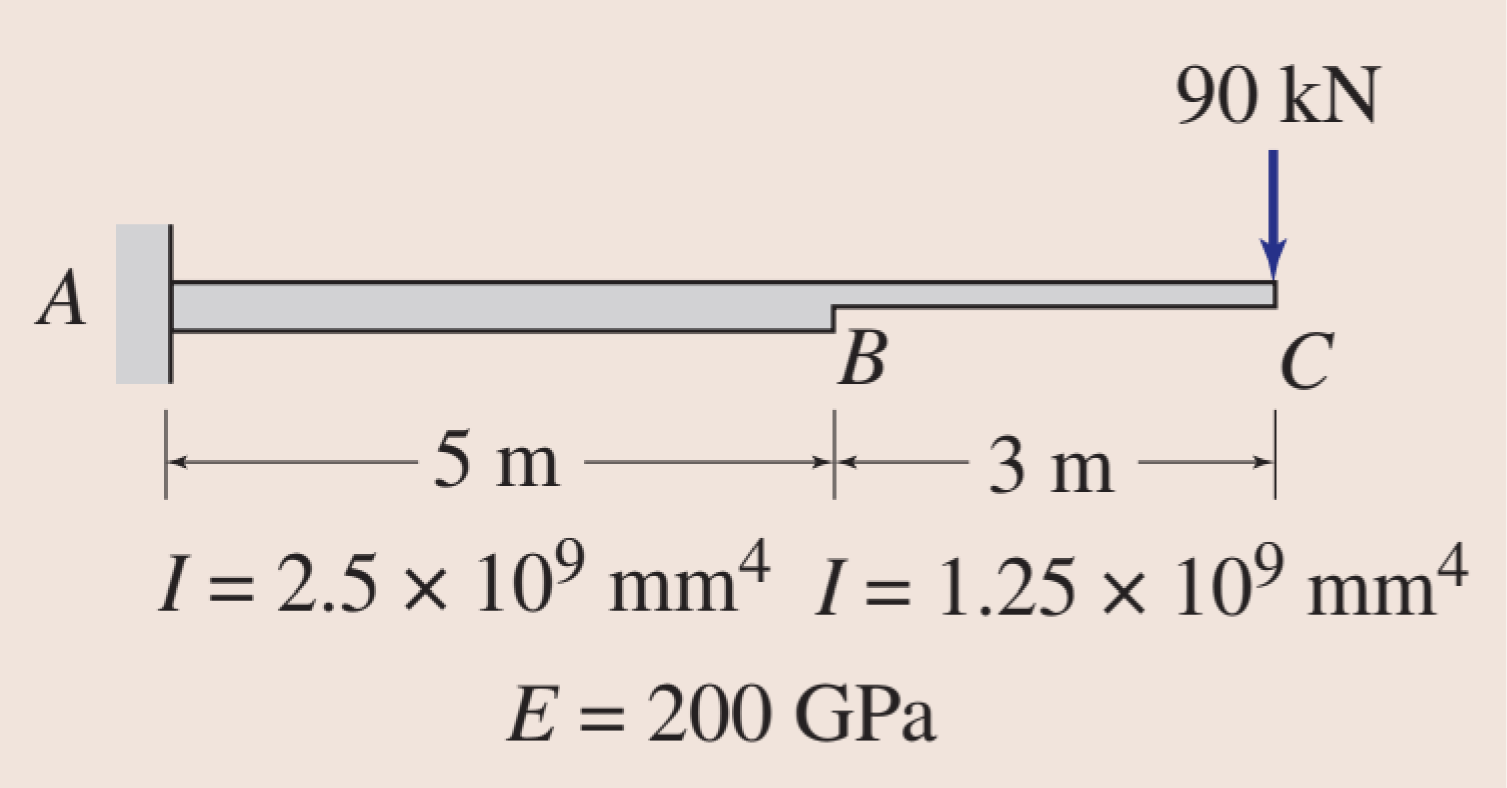
Substituting numerical values for EI, we have

$$\theta_C = \frac{-900\ \text{k}\cdot\text{ft}^2(144\ \text{in}^2/\text{ft}^2)}{29(10^3)\ \text{k/in}^2(600\ \text{in}^4)}$$

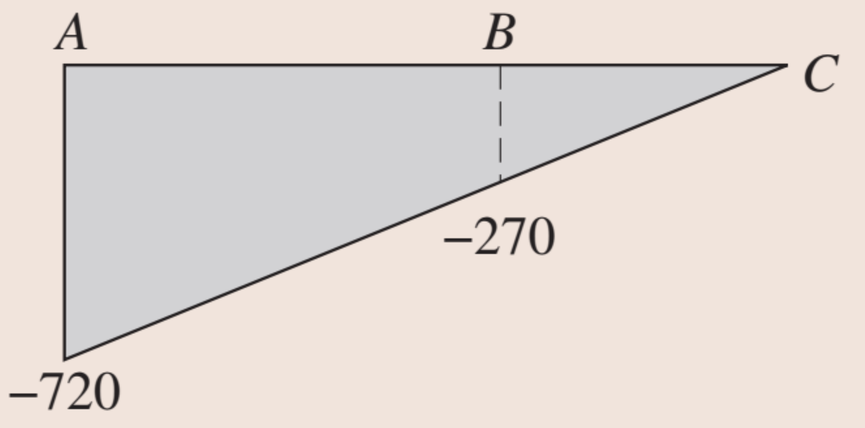
$$= -0.00745\ \text{rad} \qquad \textit{Ans.}$$

Example 3.4

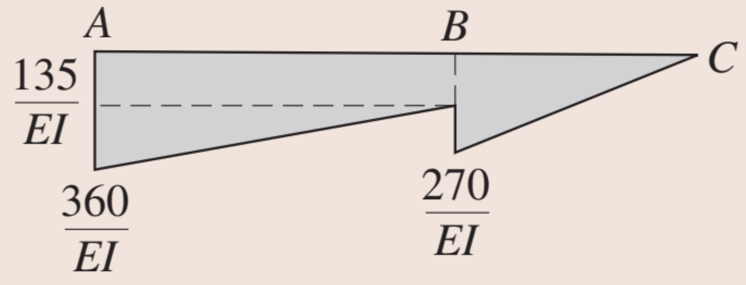
Determine the slopes and deflections at points *B* and *C* of the cantilever beam shown in the figure by the moment-area method.



Solution 3.4



(b) Bending Moment Diagram (kN · m)



(c) $\frac{M}{EI}$ Diagram $\left(\frac{\text{kN} \cdot \text{m}}{EI}\right.$ with $I = 1.25 \times 10^9$ mm$^4\Big)$

M/EI Diagram. As indicated in Fig. 6.5(a), the values of the moment of inertia of the segments AB and BC of the beam are 2.5×10^9 mm^4 and 1.25×10^9 mm^4, respectively. Using $I = I_{BC} = 1.25 \times 10^9$ mm^4 as the reference moment of inertia, we express I_{AB} in terms of I as

$$I_{AB} = 2.5 \times 10^9 \text{ mm}^4 = 2(1.25 \times 10^9 \text{ mm}^4) = 2I$$

which indicates that in order to obtain the M/EI diagram in terms of EI, we must divide the bending moment diagram for segment AB by 2, as shown in Fig. 6.5(c).

Solution 3.4

Elastic Curve. The elastic curve for the beam is shown in Fig. 6.5(d). Note that because the M/EI diagram is negative, the beam bends concave downward. Since the support at A is fixed, the slope at A is zero ($\theta_A = 0$); that is, the tangent to the elastic curve at A is horizontal, as shown in the figure.

Slope at B. With the slope at A known, we can determine the slope at B by evaluating the change in slope θ_{BA} between A and B (which is the angle between the tangents to the elastic curve at points A and B, as shown in Fig. 6.5(d)). According to the first moment-area theorem, θ_{BA} = area of the M/EI diagram between A and B. This area can be conveniently evaluated by dividing the M/EI diagram into triangular and rectangular parts, as shown in Fig. 6.5(c). Thus,

$$\theta_{BA} = \frac{1}{EI}\left[(135)(5) + \frac{1}{2}(225)(5)\right] = \frac{1{,}237.5\,\text{kN}\cdot\text{m}^2}{EI}$$

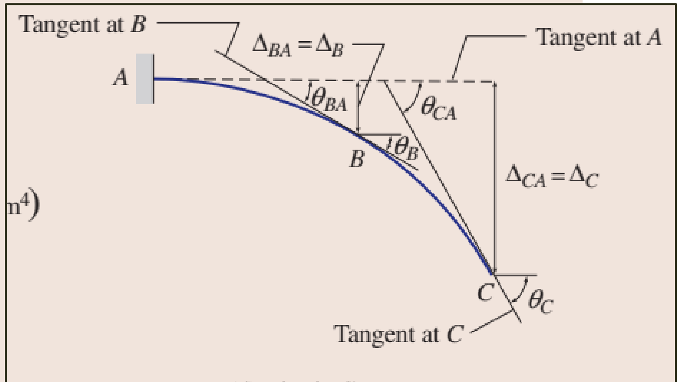
From Fig. 6.5(d), we can see that because the tangent at A is horizontal (in the direction of the undeformed axis of the beam), the slope at $B(\theta_B)$ is equal to the angle θ_{BA} between the tangents at A and B; that is,

$$\theta_B = \theta_{BA} = \frac{1{,}237.5\ \text{kN}\cdot\text{m}^2}{EI}$$

Substituting the numerical values of $E = 200 \times 10^6\,\text{kN/m}^2$ and $I = 1.25 \times 10^{-3}\,\text{m}^4$, we obtain

$$\theta_B = \frac{1{,}237.5}{(200\times10^6)(1.25\times10^{-3})}\ \text{rad} = 0.0049\ \text{rad}$$

$$\theta_B = 0.0049\ \text{rad}$$



(d) Elastic Curve

Solution 3.4

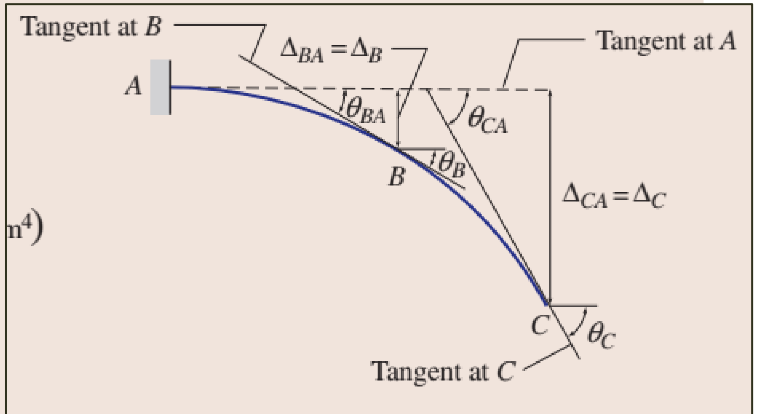
Deflection at *B*. From Fig. 6.5(d), it can be seen that the deflection of *B* with respect to the undeformed axis of the beam is equal to the tangential deviation of *B* from the tangent at *A*; that is,

$$\Delta_B = \Delta_{BA}$$

According to the second moment-area theorem,

$$\Delta_{BA} = \text{moment of the area of the } M/EI \text{ diagram between } A \text{ and } B \text{ about } B$$

$$= \frac{1}{EI}\left[(135)(5)(2.5) + \frac{1}{2}(225)(5)\left(\frac{10}{3}\right)\right] = \frac{3562.5\ \text{kN}\cdot\text{m}^3}{EI}$$



(d) Elastic Curve

Therefore,

$$\Delta_B = \Delta_{BA} = \frac{3562.5\ \text{kN}\cdot\text{m}^3}{EI}$$

$$= \frac{3562.5}{(200\times 10^6)(1.25\times 10^{-3})} = 0.01425\ \text{m}$$

$$\Delta_B = 14.25\ \text{mm}\downarrow$$

Ans.

Solution 3.4

Slope at *C*. From Fig. 6.5(d), we can see that

$$\theta_C = \theta_{CA}$$

where

$$\theta_{CA} = \text{area of the } M/EI \text{ diagram between } A \text{ and } C$$

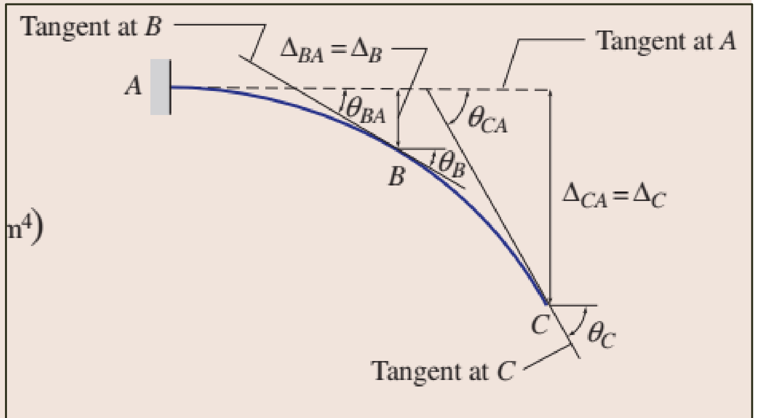
$$= \frac{1}{EI}\left[(135)(5) + \frac{1}{2}(225)(5) + \frac{1}{2}(270)(3)\right] = \frac{1642.5 \text{ kN}\cdot\text{m}^2}{EI}$$

Therefore,

$$\theta_C = \theta_{CA} = \frac{1642.5 \text{ kN}\cdot\text{m}^2}{EI}$$

$$= \frac{1642.5}{(200\times10^6)(1.25\times10^{-3})} = 0.0066 \text{ rad}$$

$$\theta_C = 0.0066 \text{ rad}$$



(d) Elastic Curve

Solution 3.4

Deflection at *C*. It can be seen from Fig. 6.5(d) that

$$\Delta_C = \Delta_{CA}$$

where

$$\Delta_{CA} = \text{moment of the area of the } M/EI \text{ diagram between } A \text{ and } C \text{ about } C$$

$$= \frac{1}{EI}\left[(135)(5)(2.5+3) + \frac{1}{2}(225)(5)\left(\frac{10}{3}+3\right) + \frac{1}{2}(270)(3)(2)\right]$$

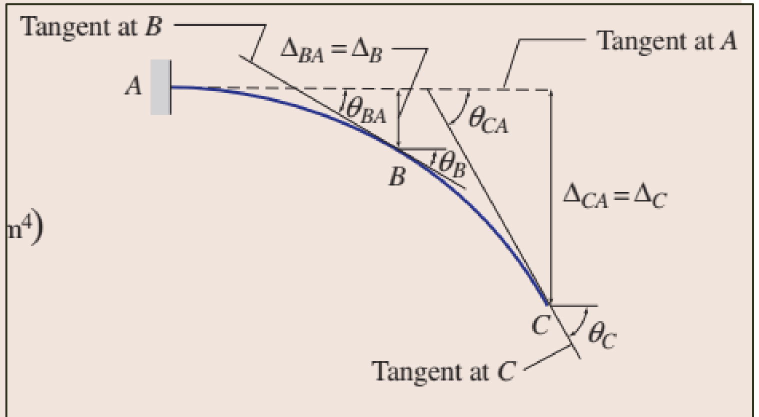
$$= \frac{8085 \text{ kN}\cdot\text{m}^3}{EI}$$

Therefore,

$$\Delta_C = \Delta_{CA} = \frac{8085 \text{ kN}\cdot\text{m}^3}{EI}$$

$$= \frac{8085}{(200\times10^6)(1.25\times10^{-3})} = 0.032 \text{ m}$$

$$\Delta_C = 32\,\text{mm} \downarrow$$

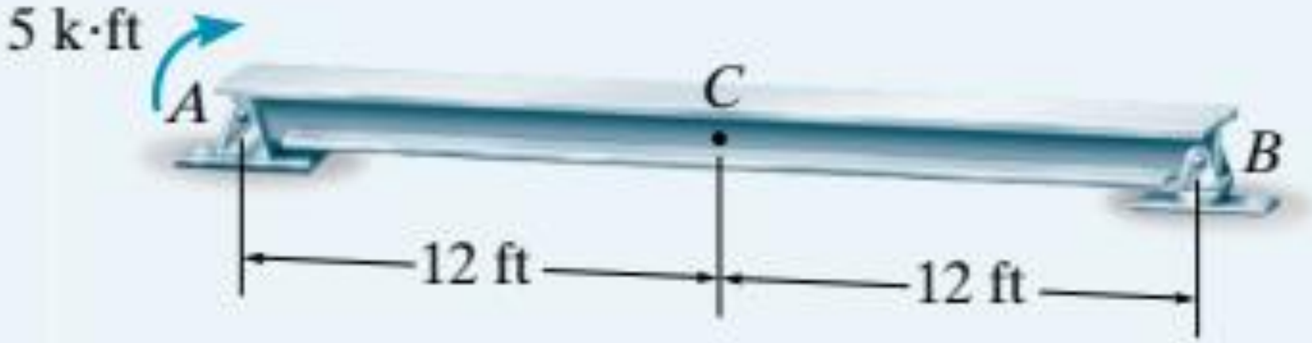


(d) Elastic Curve

Ans.

Example 3.5

Determine the deflection at C of the beam shown in Fig. 8–20a. Take $E = 29(10^3)$ ksi, $I = 21\text{ in}^4$.



Solution 3.5

M/EI Diagram. Fig. 8–20*b*.

Elastic Curve. Here we are required to find Δ_C, Fig. 8–20*c*. This is not necessarily the maximum deflection of the beam, since the loading and hence the elastic curve are *not symmetric*. Also indicated in Fig. 8–20*c* are the tangents at A, B (the supports), and C. If $t_{A/B}$ is determined, then Δ' can be found from proportional triangles, that is, $\Delta'/12 = t_{A/B}/24$ or $\Delta' = t_{A/B}/2$. From the construction in Fig. 8–20*c*, we have

$$\Delta_C = \frac{t_{A/B}}{2} - t_{C/B} \qquad (1)$$

Moment-Area Theorem. We will apply Theorem 2 to determine $t_{A/B}$ and $t_{C/B}$. Here $t_{A/B}$ is the moment of the M/EI diagram between A and B about point A,

$$t_{A/B} = \left[\frac{1}{3}(24\text{ ft})\right]\left[\frac{1}{2}(24\text{ ft})\left(\frac{5\text{ k}\cdot\text{ft}}{EI}\right)\right] = \frac{480\text{ k}\cdot\text{ft}^3}{EI}$$

and $t_{C/B}$ is the moment of the M/EI diagram between C and B about C.

$$t_{C/B} = \left[\frac{1}{3}(12\text{ ft})\right]\left[\frac{1}{2}(12\text{ ft})\left(\frac{2.5\text{ k}\cdot\text{ft}}{EI}\right)\right] = \frac{60\text{ k}\cdot\text{ft}^3}{EI}$$

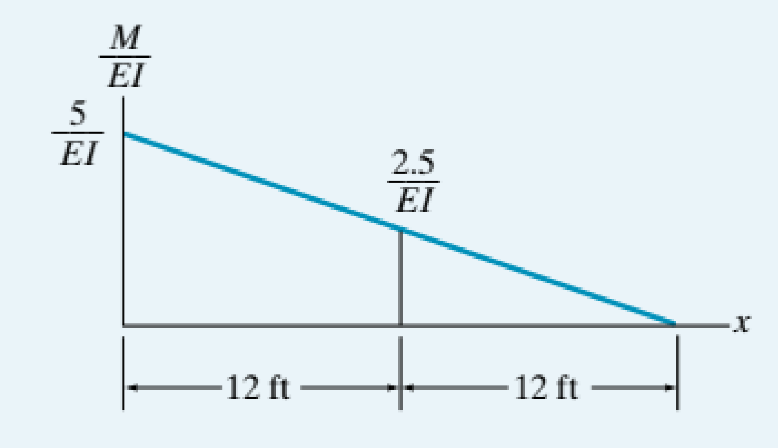
Substituting these results into Eq. (1) yields

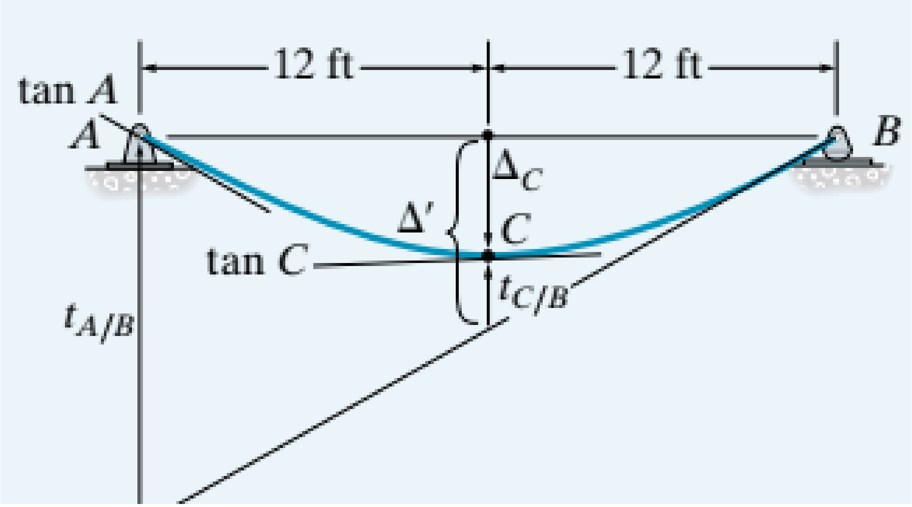
$$\Delta_C = \frac{1}{2}\left(\frac{480\text{ k}\cdot\text{ft}^3}{EI}\right) - \frac{60\text{ k}\cdot\text{ft}^3}{EI} = \frac{180\text{ k}\cdot\text{ft}^3}{EI}$$

Working in units of kips and inches, we have

$$\Delta_C = \frac{180\text{ k}\cdot\text{ft}^3(1728\text{ in}^3/\text{ft}^3)}{29(10^3)\text{ k/in}^2(21\text{ in}^4)}$$

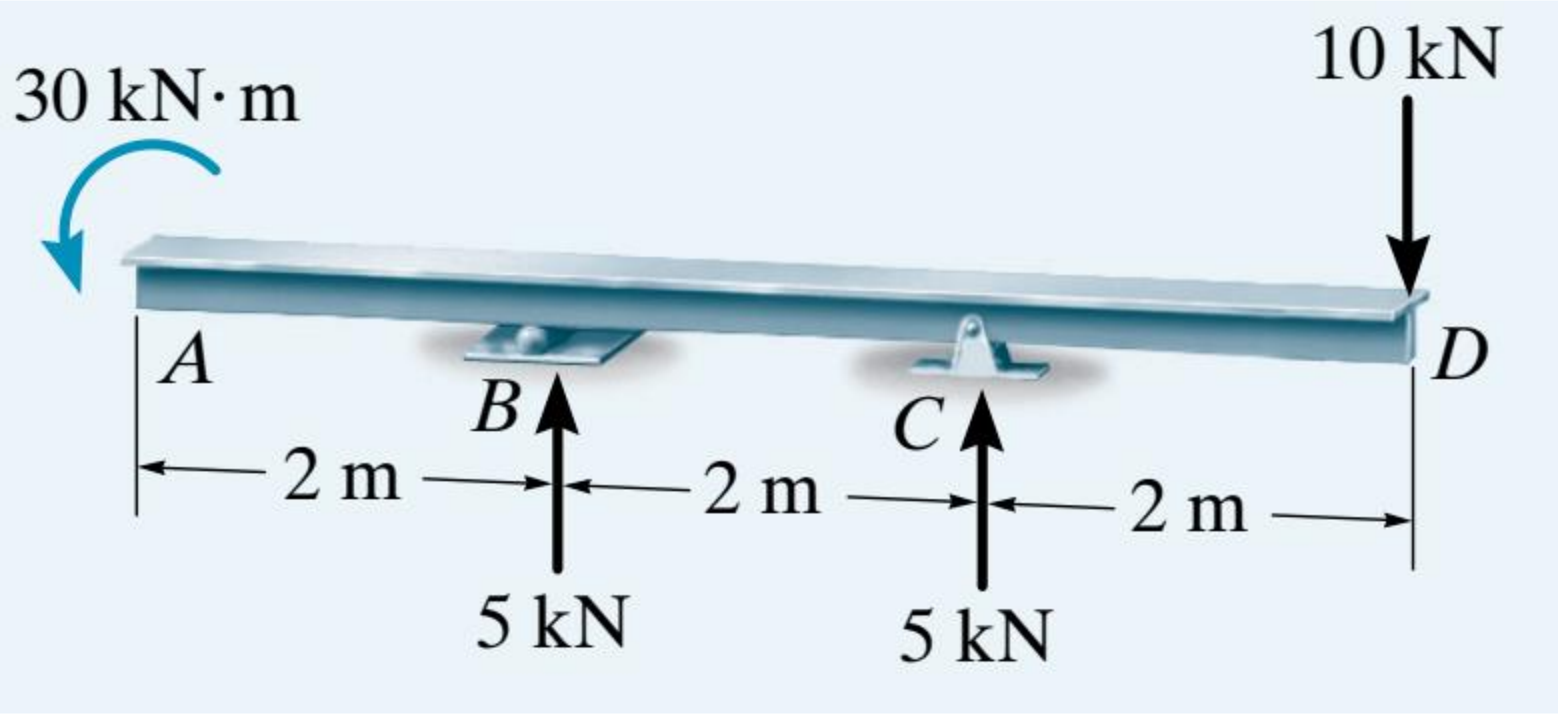
$$= 0.511\text{ in.} \qquad \textit{Ans.}$$





Example 3.6

Determine the slope at the roller B of the double overhang beam shown in Fig. 8–22a. Take E = 200 GPa, $I = 18(10^6)$ mm^4.



Solution 3.6

M/EI Diagram. The M/EI diagram can be simplified by drawing it in parts as discussed in Sec. 4.5. Here we will consider the M/EI diagrams for the three loadings each acting on a cantilever beam fixed at D, Fig. 8–22*b*. (The 10-kN load is not considered since it produces no moment about D.)

Elastic Curve. If tangents are drawn at B and C, Fig. 8–22*c*, the slope B can be determined by finding $t_{C/B}$, and for small angles,

$$\theta_B = \frac{t_{C/B}}{2\text{ m}} \qquad (1)$$

Moment Area Theorem. To determine $t_{C/B}$ we apply the moment area theorem by finding the moment of the M/EI diagram between BC about point C. This only involves the shaded area under two of the diagrams in Fig. 8–22*b*. Thus,

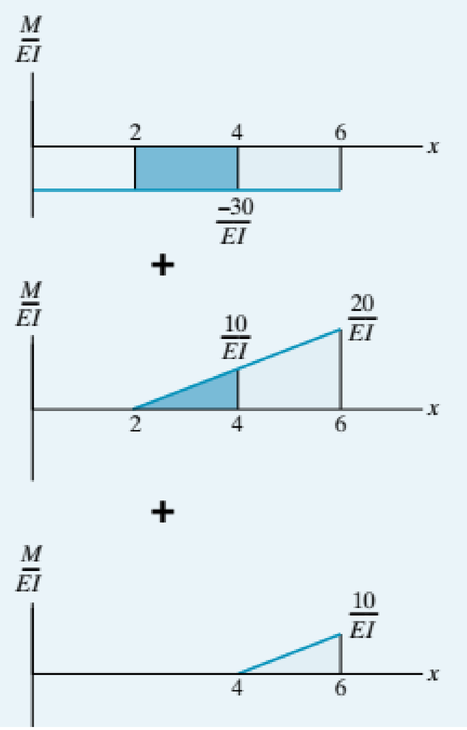
$$t_{C/B} = (1\text{ m})\left[(2\text{ m})\left(\frac{-30\text{ kN}\cdot\text{m}}{EI}\right)\right] + \left(\frac{2\text{ m}}{3}\right)\left[\frac{1}{2}(2\text{ m})\left(\frac{10\text{ kN}\cdot\text{m}}{EI}\right)\right]$$

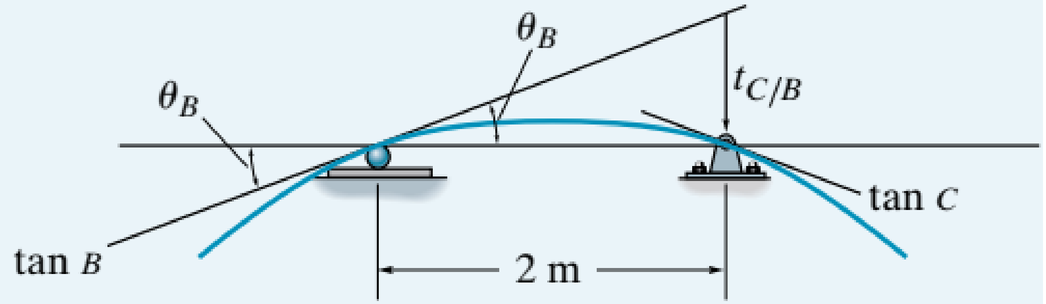
$$= -\frac{53.33\text{ kN}\cdot\text{m}^3}{EI}$$

Substituting the positive value into Eq. (1),

$$\theta_B = \frac{53.33\text{ kN}\cdot\text{m}^3}{(2\text{ m})\left[200(10^6)\text{ kN/m}^3\right]\left[18(10^6)(10^{-12})\text{ m}^4\right]}$$

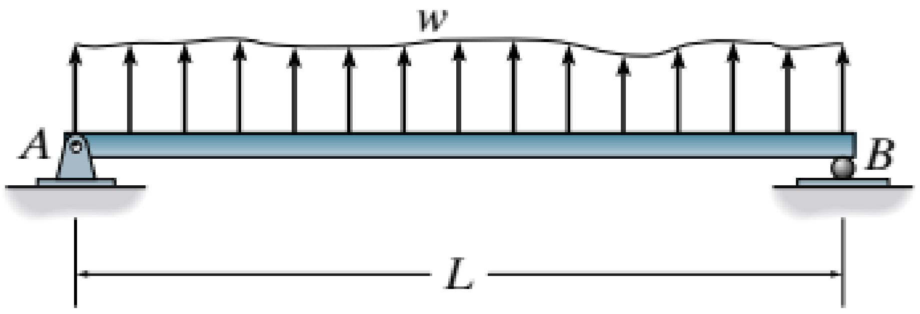
$$= 0.00741\text{ rad} \qquad \textit{Ans.}$$



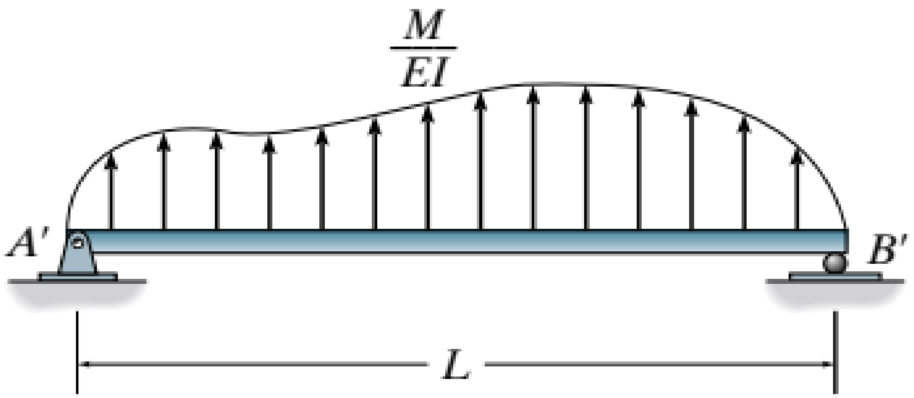


Conjugate Beam Method

Conjugate Beam Method



real beam



conjugate beam

- This method was developed by Muller Breslau and relies only on the principles of statics.

$$V = \int w\,dx \qquad M = \int\left[\int w\,dx\right]dx$$

$$\updownarrow \qquad \updownarrow \qquad \updownarrow \qquad \updownarrow$$

$$\theta = \int\left(\frac{M}{EI}\right)dx \qquad v = \int\left[\int\left(\frac{M}{EI}\right)dx\right]dx$$

- Shear V compares with the slope θ, the moment M compares with the displacement v, and the external load w compares with the M/EI diagram.
- Thus a beam having the same length as the real beam, but referred to here as the "conjugate beam,"
- M/EI diagram derived from the load w on the real beam.

Conjugate Beam Method

- **Theorem 1: The slope at a point in the real beam is numerically equal to the shear at the corresponding point in the conjugate beam.**
- **Theorem 2: The displacement of a point in the real beam is numerically equal to the moment at the corresponding point in the conjugate beam.**
- The shear and moment developed at the supports of the conjugate beam account for the corresponding slope and displacement of the real beam at its supports
- neglecting axial force, statically determinate real beams have statically determinate conjugate beams; and statically indeterminate real beams

	Real Beam		Conjugate Beam	
1)	θ, $\Delta = 0$	pin	V, $M = 0$	pin
2)	θ, $\Delta = 0$	roller	V, $M = 0$	roller
3)	$\theta = 0$, $\Delta = 0$	fixed	$V = 0$, $M = 0$	free
4)	θ, Δ	free	V, M	fixed
5)	θ, $\Delta = 0$	internal pin	V, $M = 0$	hinge
6)	θ, $\Delta = 0$	internal roller	V, $M = 0$	hinge
7)	θ, Δ	hinge	V, M	internal roller

Conjugate Beam Method

Procedure for Analysis

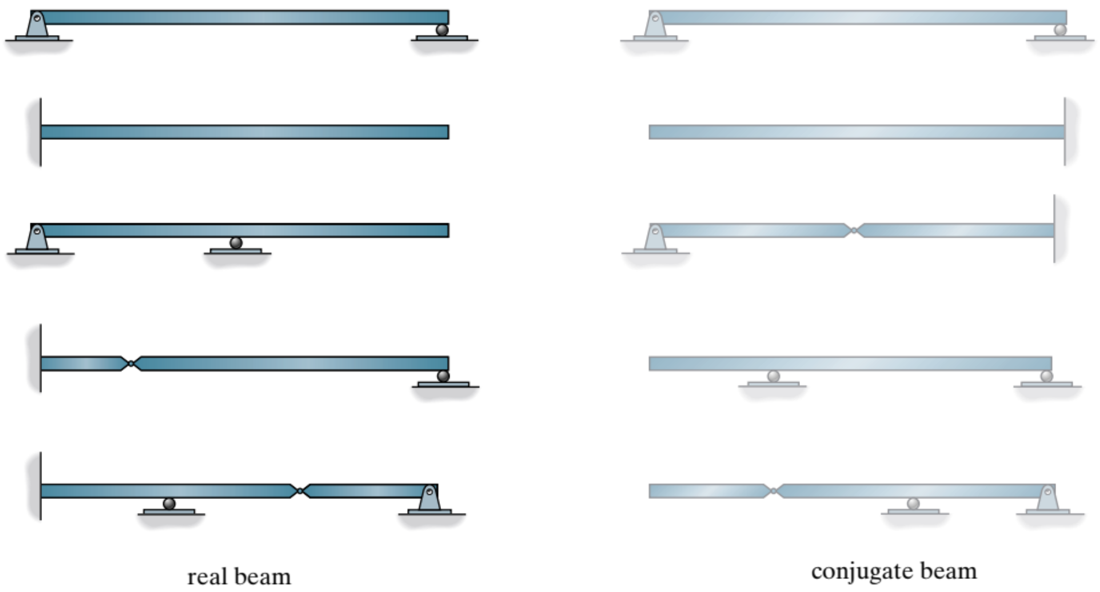
The following procedure provides a method that may be used to determine the displacement and slope at a point on the elastic curve of a beam using the conjugate-beam method.

Conjugate Beam

- Draw the conjugate beam for the real beam. This beam has the same length as the real beam and has corresponding supports as listed in Table 8.2.
- In general, if the real support allows a *slope,* the conjugate support must develop a *shear;* and if the real support allows a *displacement,* the conjugate support must develop a *moment*.
- The conjugate beam is loaded with the real beam's M/EI diagram. This loading is assumed to be *distributed* over the conjugate beam and is directed *upward* when M/EI is *positive* and *downward* when M/EI is *negative*. In other words, the loading always acts *away* from the beam.

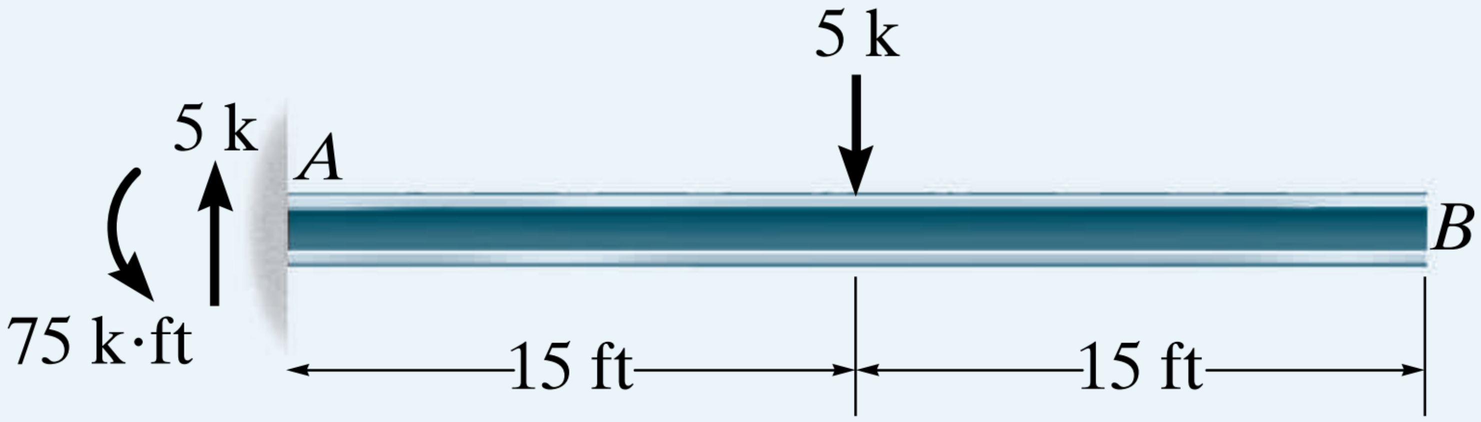
Equilibrium

- Using the equations of equilibrium, determine the reactions at the conjugate beam's supports.
- Section the conjugate beam at the point where the slope θ and displacement Δ of the real beam are to be determined. At the section show the unknown shear V' and moment M' acting in their positive sense.
- Determine the shear and moment using the equations of equilibrium. V' and M' equal θ and Δ, respectively, for the real beam. In particular, if these values are *positive,* the *slope* is *counterclockwise* and the *displacement* is *upward*.



Example 3.7

Determine the slope and deflection at point B of the steel beam shown in Fig. 8–25a. The reactions have been computed. $E = 29(10^3)$ ksi, $I = 800$ in^4.



real beam
(a)

Example 3.7

Conjugate Beam. The conjugate beam is shown in Fig. 8–25*b*. The supports at A' and B' correspond to supports A and B on the real beam, Table 8.2. It is important to understand why this is so. The M/EI diagram is *negative*, so the distributed load acts *downward*, i.e., away from the beam.

Equilibrium. Since θ_B and Δ_B are to be determined, we must compute $V_{B'}$ and $M_{B'}$ in the conjugate beam, Fig. 8–25*c*.

$$+\uparrow\Sigma F_y = 0; \qquad -\frac{562.5\ \text{k}\cdot\text{ft}^2}{EI} - V_{B'} = 0$$

$$\theta_B = V_{B'} = -\frac{562.5\ \text{k}\cdot\text{ft}^2}{EI}$$

$$= \frac{-562.5\ \text{k}\cdot\text{ft}^2}{29(10^3)\ \text{k/in}^2(144\ \text{in}^2/\text{ft}^2)800\ \text{in}^4(1\ \text{ft}^4/(12)^4\ \text{in}^4)}$$

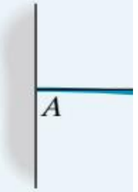
$$= -0.00349\ \text{rad} \qquad \textit{Ans.}$$

$$\circlearrowleft +\Sigma M_{B'} = 0; \qquad \frac{562.5\ \text{k}\cdot\text{ft}^2}{EI}(25\ \text{ft}) + M_{B'} = 0$$

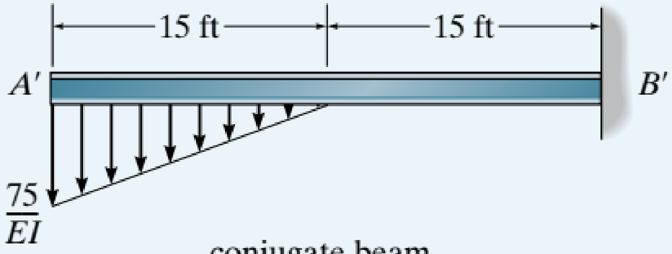
$$\Delta_B = M_{B'} = -\frac{14\,062.5\ \text{k}\cdot\text{ft}^3}{EI}$$

$$= \frac{-14\,062.5\ \text{k}\cdot\text{ft}^3}{29(10^3)(144)\ \text{k/ft}^2\left[800/(12)^4\right]\ \text{ft}^4}$$

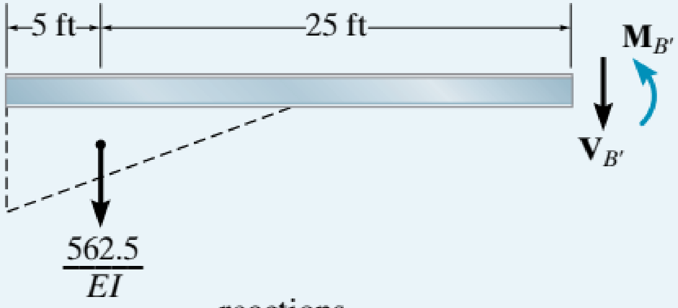
$$= -0.0873\ \text{ft} = -1.05\ \text{in.} \qquad \textit{Ans.}$$



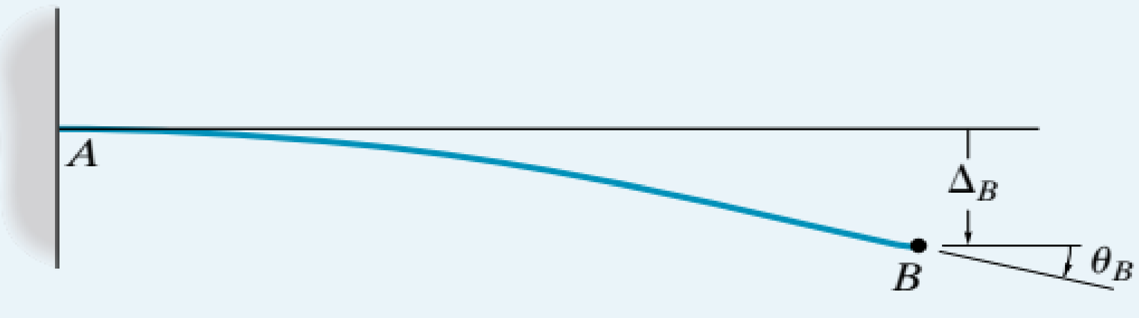
The negative signs indicate the slope of the beam is measured clockwise and the displacement is downward, Fig. 8–25*d*.



conjugate beam
(b)



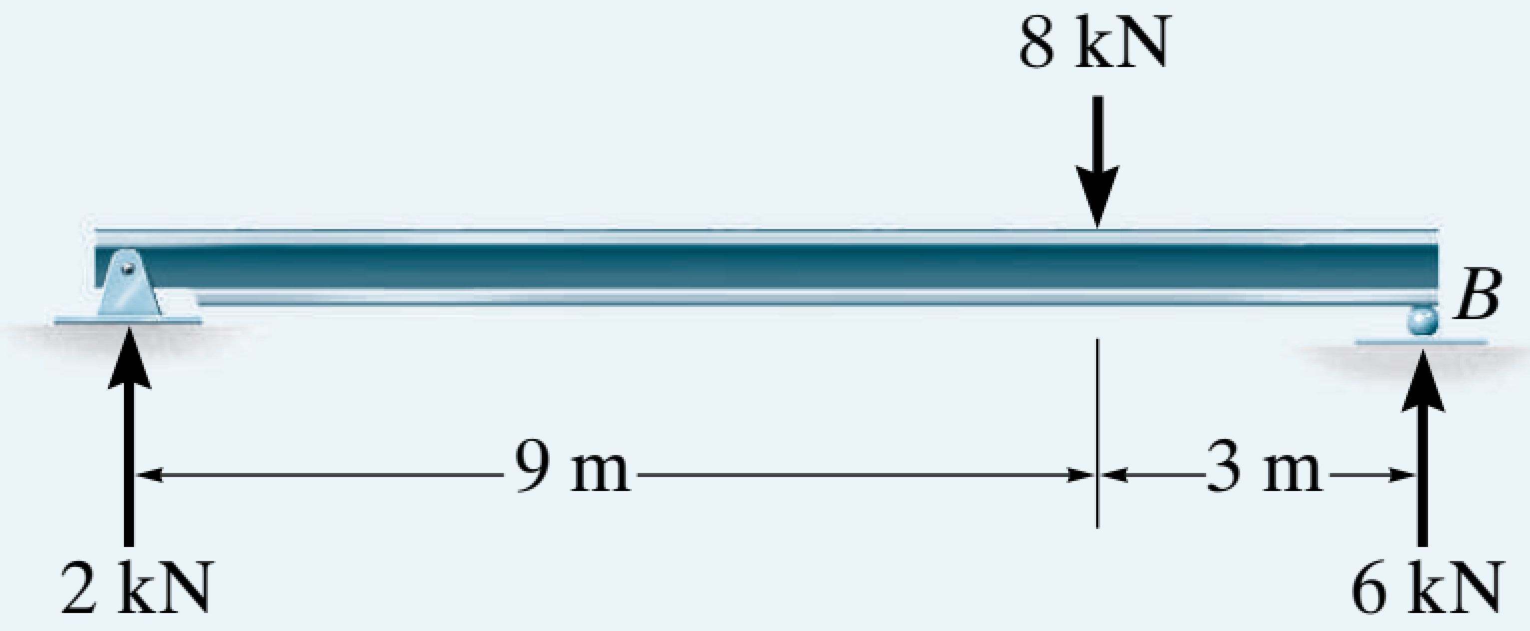
reactions
(c)



(d)

Example 3.8

Determine the maximum deflection of the steel beam shown in Fig. 8–26*a*. The reactions have been computed. $E = 200$ GPa, $I = 60(10^6)$ mm^4.



real beam

Example 3.8

Conjugate Beam. The conjugate beam loaded with the M/EI diagram is shown in Fig. 8–26*b*. Since the M/EI diagram is positive, the distributed load acts upward (away from the beam).

Equilibrium. The external reactions on the conjugate beam are determined first and are indicated on the free-body diagram in Fig. 8–26*c*. *Maximum deflection* of the real beam occurs at the point where the *slope* of the beam is *zero*. This corresponds to the same point in the conjugate beam where the *shear* is *zero*. Assuming this point acts within the region $0 \leq x \leq 9$ m from A', we can isolate the section shown in Fig. 8–26*d*. Note that the peak of the distributed loading was determined from proportional triangles, that is, $w/x = (18/EI)/9$. We require $V' = 0$ so that

$$+\uparrow \Sigma F_y = 0; \qquad -\frac{45}{EI} + \frac{1}{2}\left(\frac{2x}{EI}\right)x = 0$$

$$x = 6.71 \text{ m} \qquad (0 \leq x \leq 9 \text{ m}) \text{ OK}$$

Using this value for x, the maximum deflection in the real beam corresponds to the moment M'. Hence,

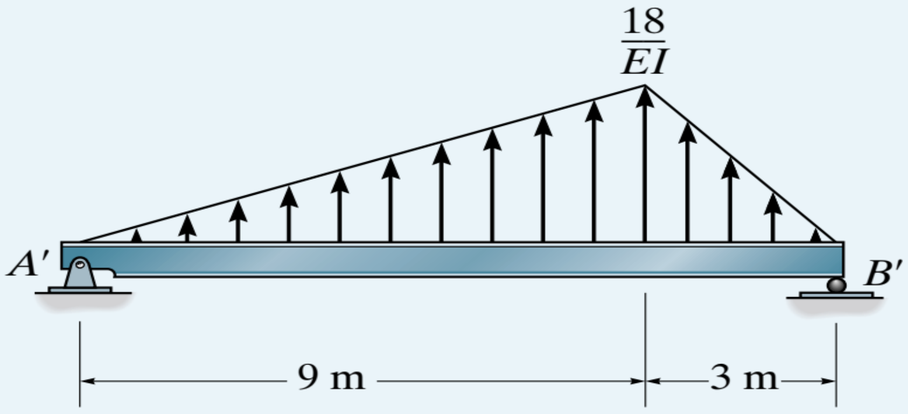
$$\circlearrowleft + \Sigma M = 0; \qquad \frac{45}{EI}(6.71) - \left[\frac{1}{2}\left(\frac{2(6.71)}{EI}\right)6.71\right]\frac{1}{3}(6.71) + M' = 0$$

$$\Delta_{max} = M' = -\frac{201.2 \text{ kN}\cdot\text{m}^3}{EI}$$

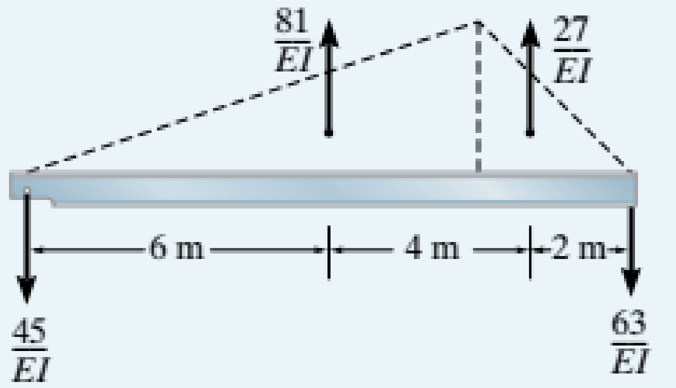
$$= \frac{-201.2 \text{ kN}\cdot\text{m}^3}{[200(10^6)\text{ kN/m}^2][60(10^6)\text{ mm}^4(1\text{ m}^4/(10^3)^4\text{ mm}^4)]}$$

$$= -0.0168 \text{ m} = -16.8 \text{ mm} \qquad \textit{Ans.}$$

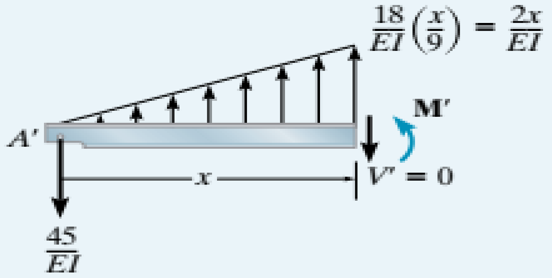
The negative sign indicates the deflection is downward.



conjugate beam



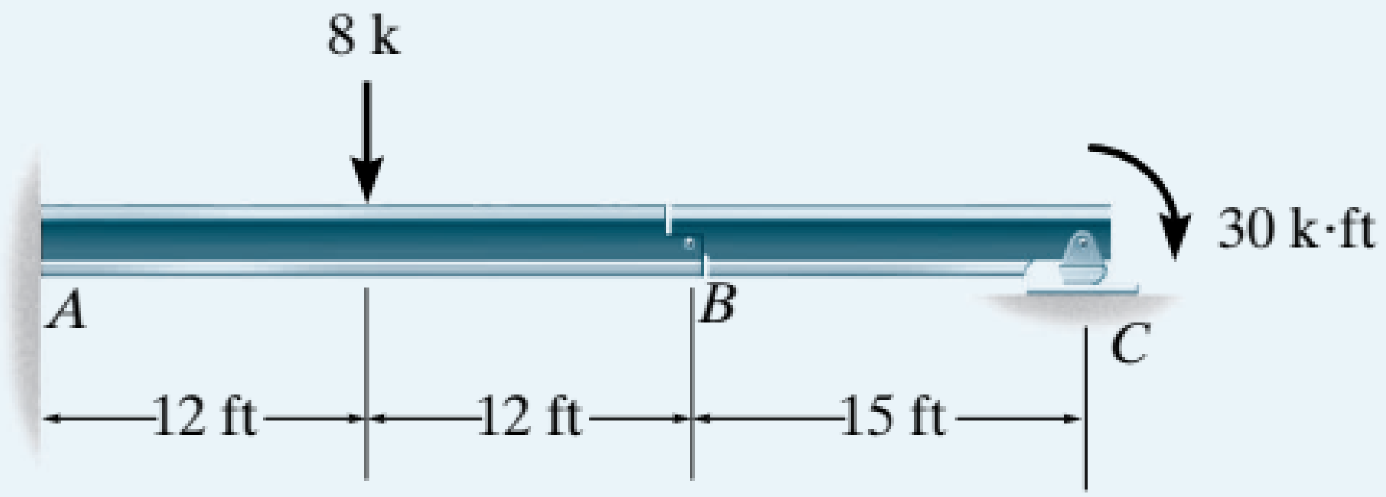
external reactions



internal reactions

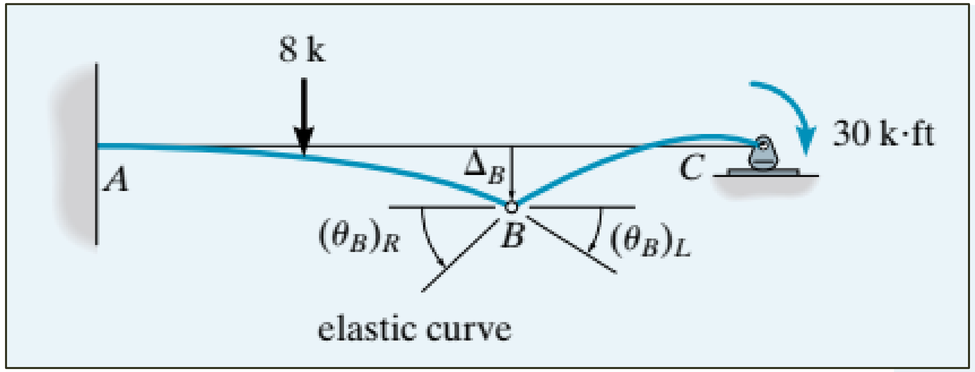
Example 3.9

Determine the displacement of the pin at B and the slope of each beam segment connected to the pin for the compound beam shown in Fig. 8–28a. $E = 29(10^3)$ ksi, $I = 30$ in^4.

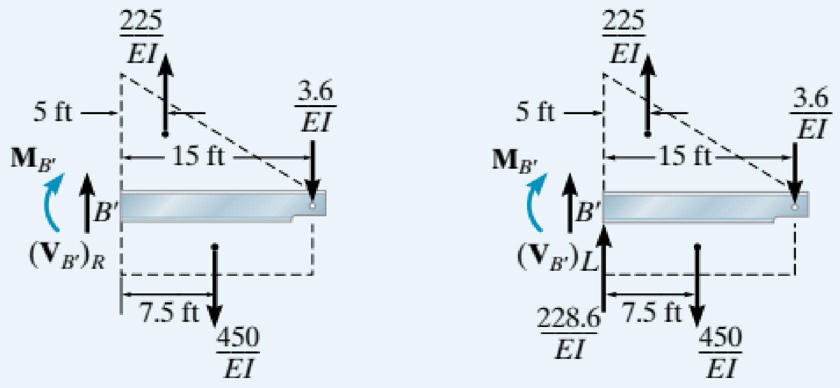


real beam

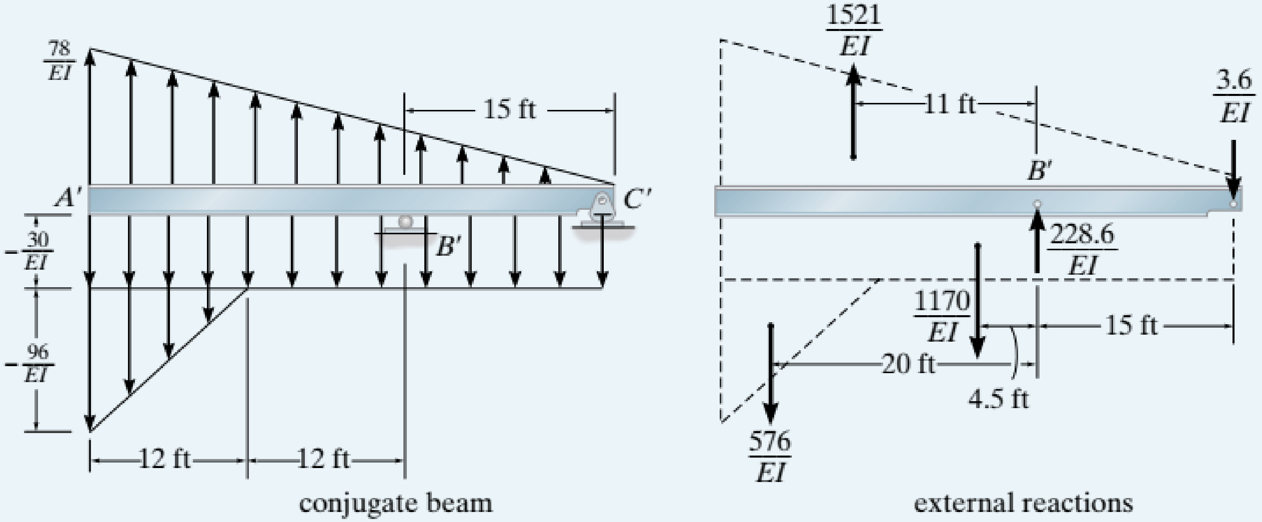
Example 3.9



Conjugate Beam. The elastic curve for the beam is shown in Fig. 8–28*b* in order to identify the unknown displacement Δ_B and the slopes $(\theta_B)_L$ and $(\theta_B)_R$ to the left and right of the pin. Using Table 8.2, the conjugate beam is shown in Fig. 8–28*c*. For simplicity in calculation, the M/EI diagram has been drawn in *parts* using the principle of superposition as described in Sec. 4.5. Here the beam is cantilevered from the left support, A. The moment diagrams for the 8-k load, the reactive force $C_y = 2$ k, and the 30-k·ft loading are given. Notice that negative regions of this diagram develop a downward distributed load and positive regions have a distributed load that acts upward.



(e) (f)



(c) (d)

Example 3.9

Equilibrium. The external reactions at B' and C' are calculated first and the results are indicated in Fig. 8–28*d*. In order to determine $(\theta_B)_R$, the conjugate beam is sectioned just to the *right* of B' and the shear force $(V_{B'})_R$ is computed, Fig. 8–28*e*. Thus,

$$+\uparrow\Sigma F_y = 0; \qquad (V_{B'})_R + \frac{225}{EI} - \frac{450}{EI} - \frac{3.6}{EI} = 0$$

$$(\theta_B)_R = (V_{B'})_R = \frac{228.6\ \text{k}\cdot\text{ft}^2}{EI}$$

$$= \frac{228.6\ \text{k}\cdot\text{ft}^2}{\left[29(10^3)(144)\ \text{k/ft}^2\right]\left[30/(12)^4\right]\ \text{ft}^4}$$

$$= 0.0378\ \text{rad} \qquad \textit{Ans.}$$

The internal moment at B' yields the displacement of the pin. Thus,

$$\circlearrowleft +\Sigma M_{B'} = 0; \qquad -M_{B'} + \frac{225}{EI}(5) - \frac{450}{EI}(7.5) - \frac{3.6}{EI}(15) = 0$$

$$\Delta_B = M_{B'} = -\frac{2304\ \text{k}\cdot\text{ft}^3}{EI}$$

$$= \frac{-2304\ \text{k}\cdot\text{ft}^3}{\left[29(10^3)(144)\ \text{k/ft}^2\right]\left[30/(12)^4\right]\ \text{ft}^4}$$

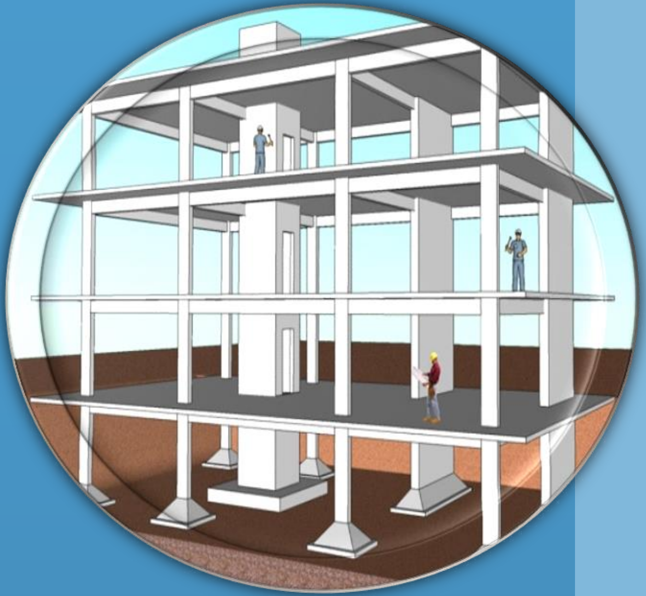
$$= -0.381\ \text{ft} = -4.58\ \text{in.} \qquad \textit{Ans.}$$

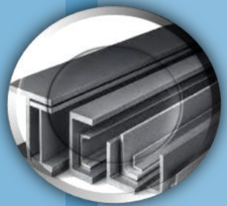
The slope $(\theta_B)_L$ can be found from a section of beam just to the *left* of B', Fig. 8–28*f*. Thus,

$$+\uparrow\Sigma F_y = 0; \qquad (V_{B'})_L + \frac{228.6}{EI} + \frac{225}{EI} - \frac{450}{EI} - \frac{3.6}{EI} = 0$$

$$(\theta_B)_L = (V_{B'})_L = 0 \qquad \textit{Ans.}$$

Obviously, $\Delta_B = M_{B'}$ for this segment is the *same* as previously calculated, since the moment arms are only slightly different in Figs. 8–28*e* and 8–28*f*.





Thank You!