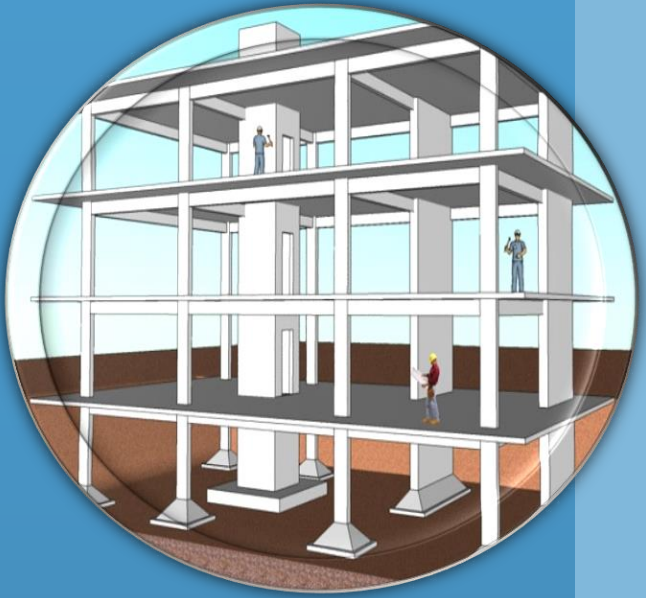
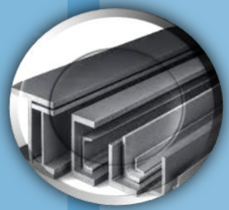
UNIVERSITY OF ZAMBIA
School of Engineering
Department of Civil & Environmental Eng.



CEE 3222: THEORY OF STRUCTURES
Lecture 3.2

ENERGY METHODS VIRTUAL WORK STRAIN ENERGY METHODS



Contents

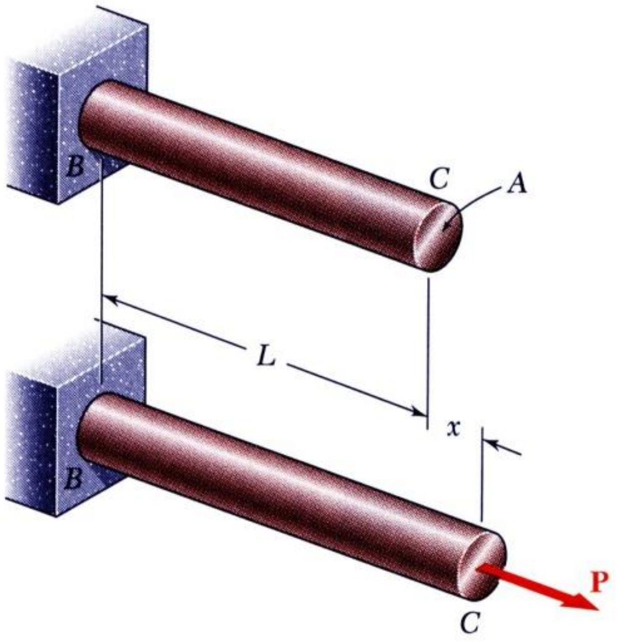
Introduction

- All energy methods are based on the conservation of energy principle, which states that the work done by all external forces acting on the structure, U_e, is transformed into internal work or strain energy. U_i, developed in the members when the structure deforms.

$$U_e = U_i$$

Strain Energy: definition

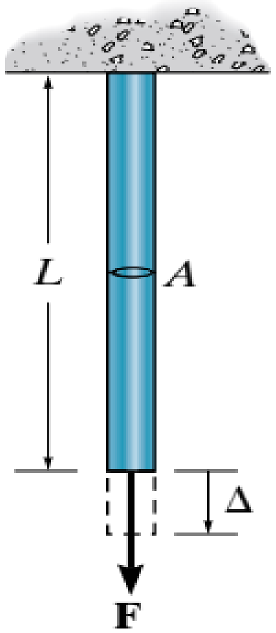
- Strain energy is a form of potential energy.
- Strain energy is defined as the increase in energy associated with the deformation of the member.
- Strain energy is equal to the work done by slowly increasing load applied to the member.
- **Work done to distort an elastic member is stored as strain energy.**
- Some energy may be lost in plastic deformation of the member and some may be converted into heat instead of stored as strain energy, but the rest is recoverable.
- A spring is an example of a storage device for strain energy.

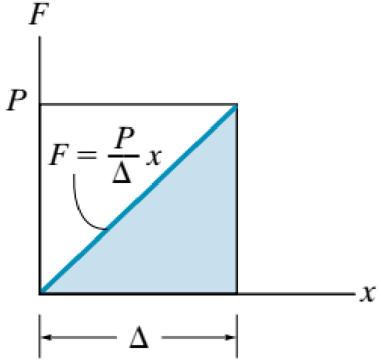


External Work and Strain Energy

- **External Work—Force**. When a force F undergoes a displacement dx in the same direction as the force, the work done is $dU_e = F dx$. If the total displacement is x, the work becomes

$$U_e = \frac{1}{2} P\Delta$$





(a)

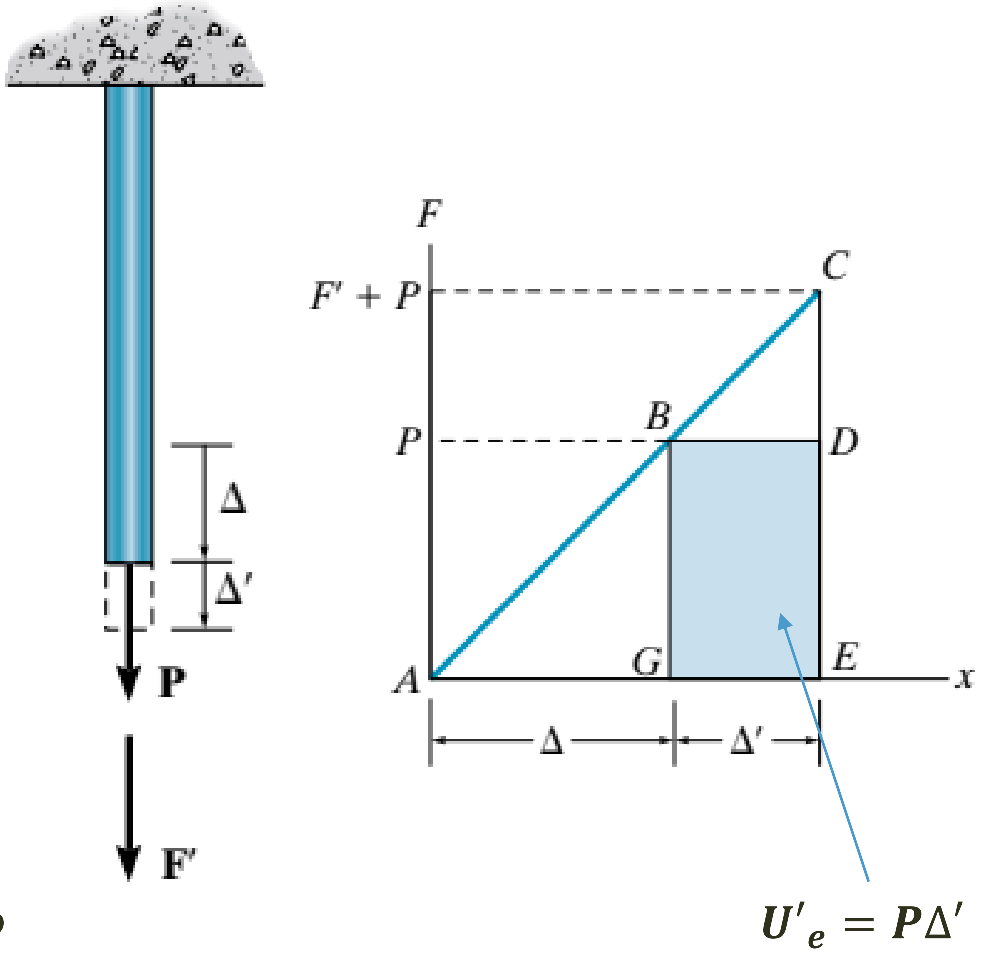
- **Strain Energy — Force**. When an axial force F is applied gradually to the bar it will strain the material such that the external work done by F will be converted into strain energy. Hookes law is obeyed for linear elastic. Thus

$$\sigma = E\epsilon \qquad \epsilon = \frac{\Delta}{L} \qquad F = P$$

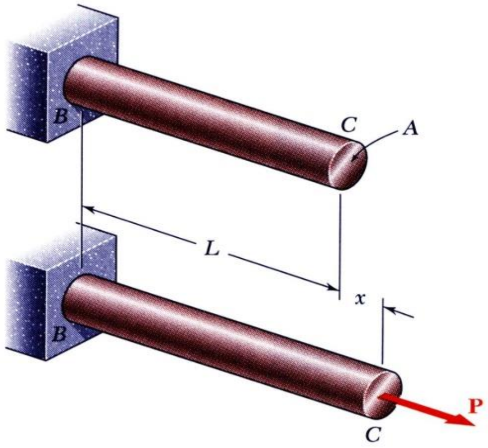
$$\sigma = \frac{F}{A} \qquad \Delta = \frac{FL}{AE} \qquad U_i = \frac{N^2 L}{2AE}$$

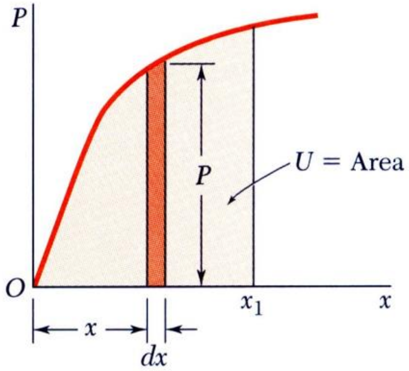
External Work and Strain Energy

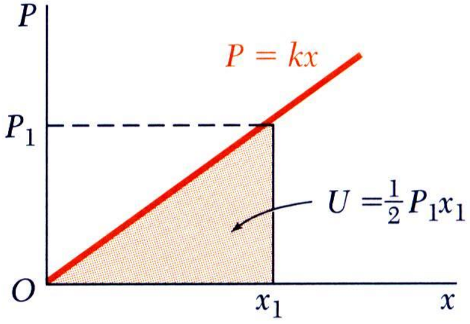
- In summary, then, when a force P is applied to the bar, followed by application of the force F', the total work done by both forces is represented by the triangular area ACE.
- The triangular area ABG represents the work of P that is caused by its displacement Δ.
- The triangular area BCD represents the work of F' since this force causes a displacement Δ'.
- The shaded rectangular area BDEG represents the additional work done by P ($\boldsymbol{U'_e = P\Delta'}$) when displaced Δ' as caused by F'.



Strain Energy Density







- **A uniform rod is subjected to a slowly increasing load**

- **The *elementary work* done by the load P as the rod elongates by a small *dx* is**

$$dU = P\,dx = elementary\ work$$

which is equal to the area of width *dx* under the load-deformation diagram.

- **The *total work* done by the load for a deformation x_1,**

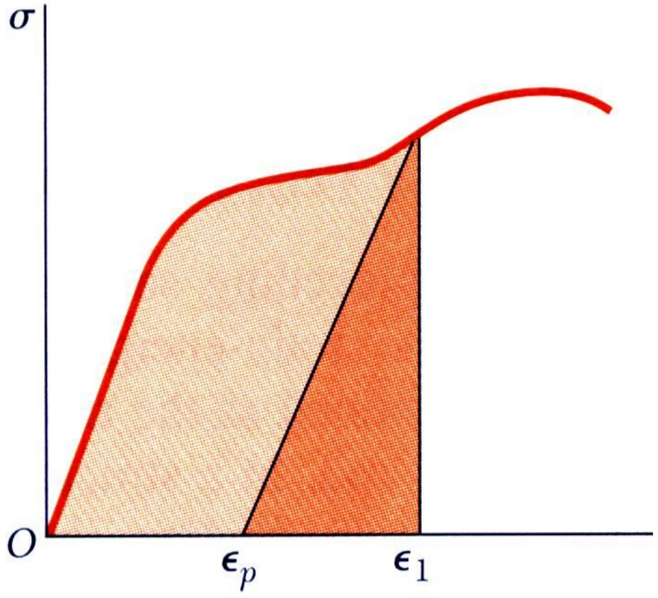
$$U = \int_0^{x_1} P\,dx = total\ work = strain\,energy$$

which results in an increase of *strain energy* in the rod.

- **In the case of a linear elastic deformation,**

$$U = \int_0^{x_1} kx\,dx = \tfrac{1}{2}kx_1^2 = \tfrac{1}{2}P_1x_1$$

Strain Energy Density



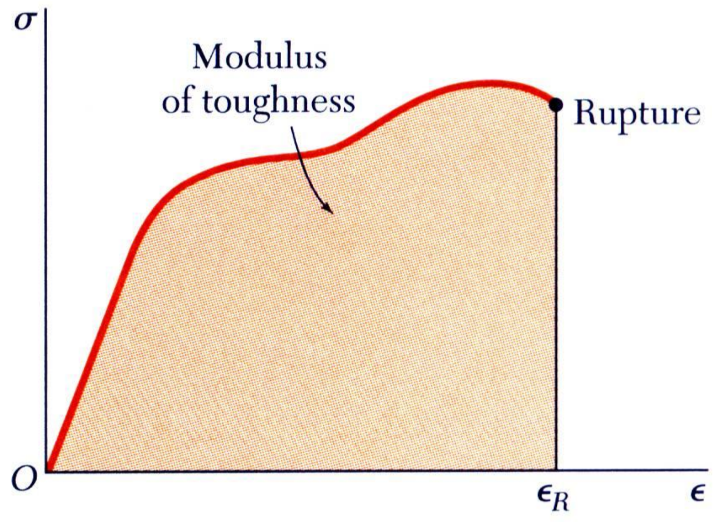
- The strain-energy density of a material is defined as the strain energy per unit volume.
- To eliminate the effects of size, evaluate the strain- energy per unit volume,

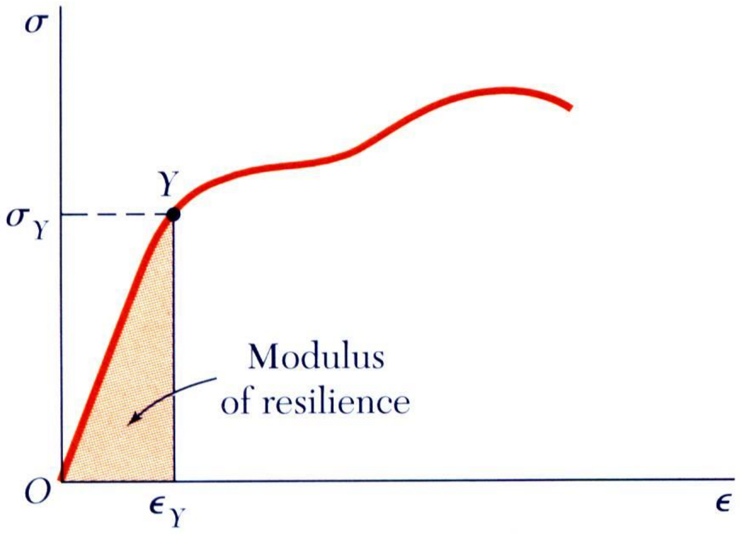
$$\frac{U}{V} = \int_0^{x_1} \frac{P}{A} \frac{dx}{L}$$

$$u = \int_0^{\varepsilon_1} \sigma_x \, d\varepsilon = strain\,energy\;density$$

- The total strain energy density resulting from the deformation is equal to the area under the curve to e_1.
- As the material is unloaded, the stress returns to zero but there is a permanent deformation. Only the strain energy represented by the triangular area is recovered.
- Remainder of the energy spent in deforming the material is dissipated as heat.

Strain-Energy Density





- The strain energy density resulting from setting $e_1 = e_R$ is the *modulus of toughness*.

- The energy per unit volume required to cause the material to rupture is related to its ductility as well as its ultimate strength.

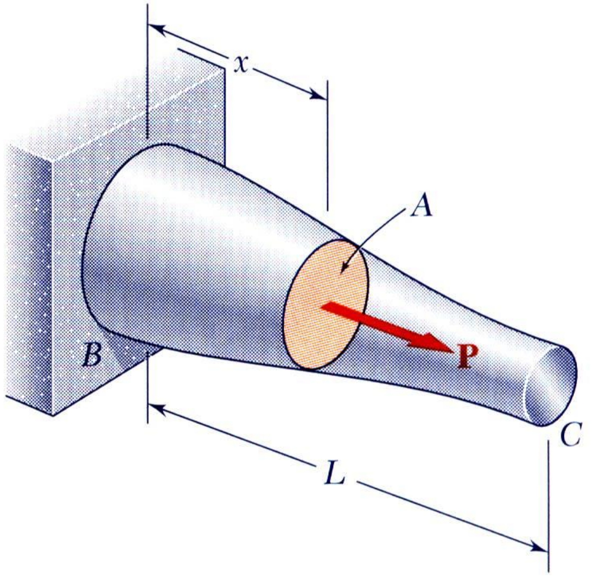
- If the stress remains within the proportional limit,

$$u = \int_0^{\varepsilon_1} E\varepsilon_1 \, d\varepsilon_x = \frac{E\varepsilon_1^2}{2} = \frac{\sigma_1^2}{2E}$$

- The strain energy density resulting from setting $s_1 = s_Y$ is the *modulus of resilience*.

$$u_Y = \frac{\sigma_Y^2}{2E} = modulus\ of\ resilience$$

Strain Energy under Axial Loading



- **In an element with a nonuniform stress distribution,**

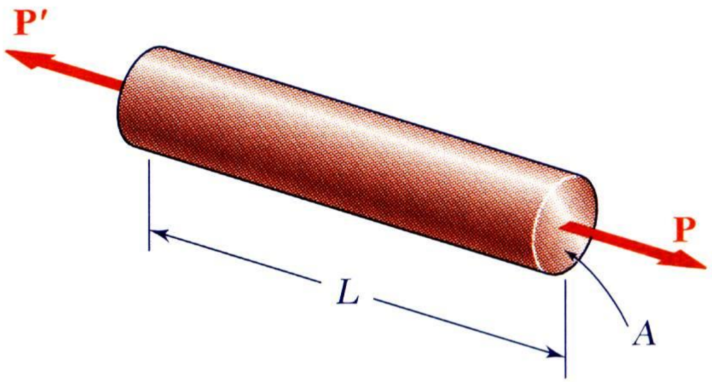
$$u = \lim_{\Delta V \to 0} \frac{\Delta U}{\Delta V} = \frac{dU}{dV} \qquad U = \int u \, dV = \text{total strain energy}$$

- **For values of $u < u_Y$, i.e., below the proportional limit,**

$$U = \int \frac{\sigma_x^2}{2E} dV = elastic\,strain\,energy$$

- **Under axial loading,** $\sigma_x = P/A \qquad dV = A\,dx$

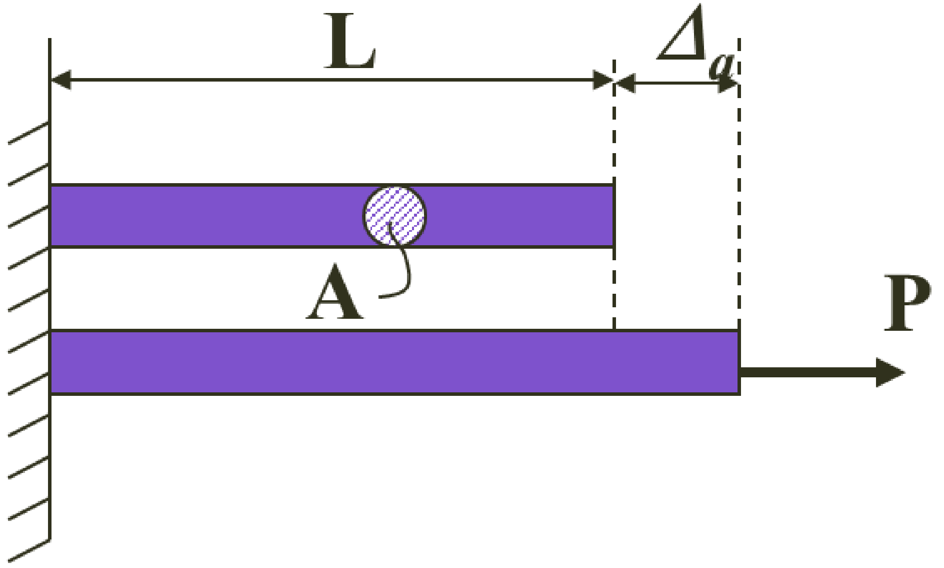
$$U = \int_0^L \frac{P^2}{2AE} dx$$

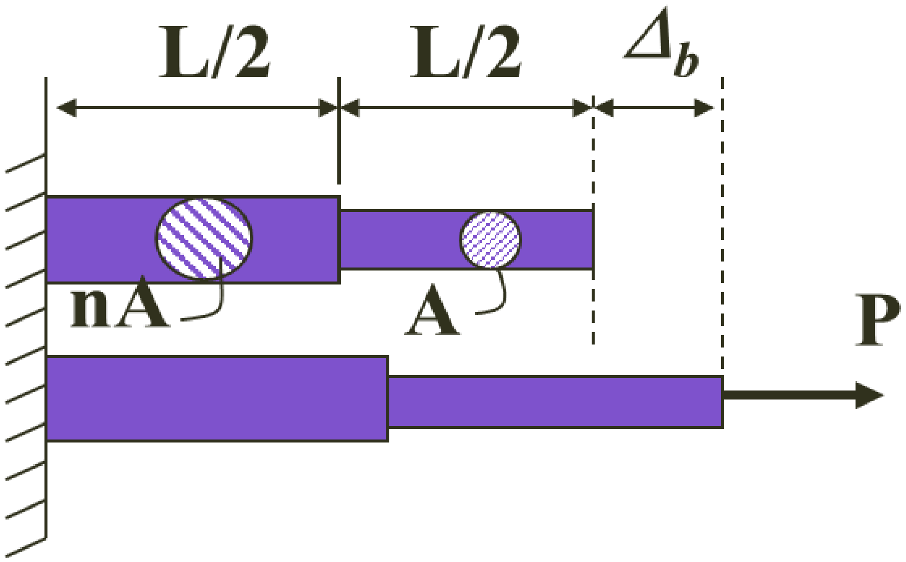


- **For a rod of uniform cross-section,**

$$U = \frac{P^2 L}{2AE}$$

Comparison of Energy Stored in Straight and Stepped bars





(a) $U = \dfrac{P^2 L}{2AE}$

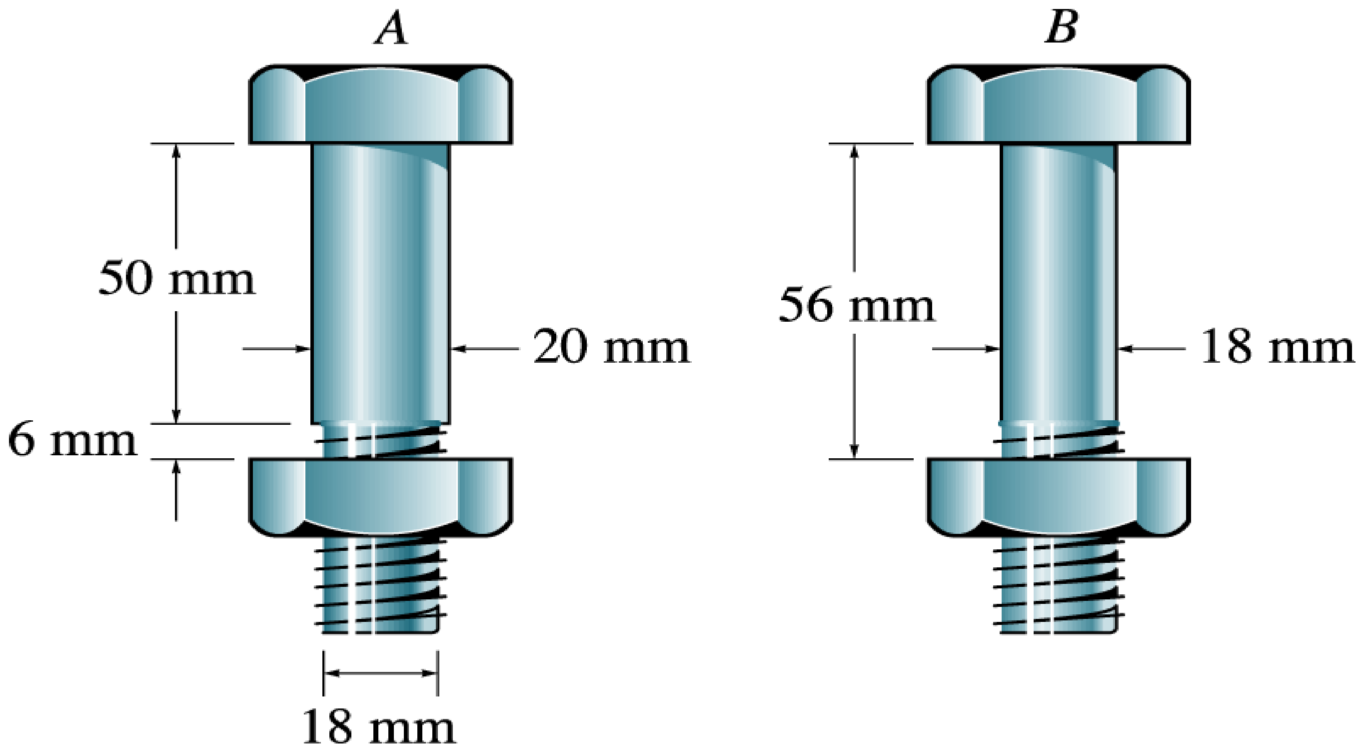
(b) $U = \dfrac{P^2 L/2}{2AE} + \dfrac{P^2 L/2}{2nAE}$

$= \dfrac{P^2 L}{2AE}\left(\dfrac{1+n}{2n}\right)$

Note for n=2; case (b) has U= $\dfrac{3}{4}\dfrac{P^2 L}{2AE}$ **which is 3/4 of case (a)**

Example 1

One of the two bolts need to support a sudden tensile loading. To choose it is necessary to determine the greatest amount of strain energy each bolt can absorb. Bolt A has a diameter of 20 mm for 50 mm length and a root diameter of 18 mm for 6 mm length. Bolt B has 18 mm diameter throughout the length. Take E = 210 GPa and σ_y= 310 Mpa.



Solution

Bolt A. If the bolt is subjected to its maximum tension, the maximum stress of $\sigma_Y = 310\ \text{N/mm}^2$ will occur within the 6 mm region. This tension force is

$$P_{\max} = \sigma_Y A = 310\ \text{N/mm}^2\left[\pi\left(\frac{18\ \text{mm}}{2}\right)^2\right] = 78886\ \text{N} = 78.89\ \text{kN}$$

Applying Eq. 14–16 to each region of the bolt, we have

$$U_i = \sum \frac{N^2L}{2AE}$$

$$= \frac{(78.89\times10^3\ \text{N})^2(50\ \text{mm})}{2[\pi(20\ \text{mm}/2)^2][210(10^3)\ \text{N/mm}^2]} + \frac{(78.89\times10^3\ \text{N})^2(6\ \text{mm})}{2[\pi(18\ \text{mm}/2)^2][210(10^3)\ \text{N/mm}^2]}$$

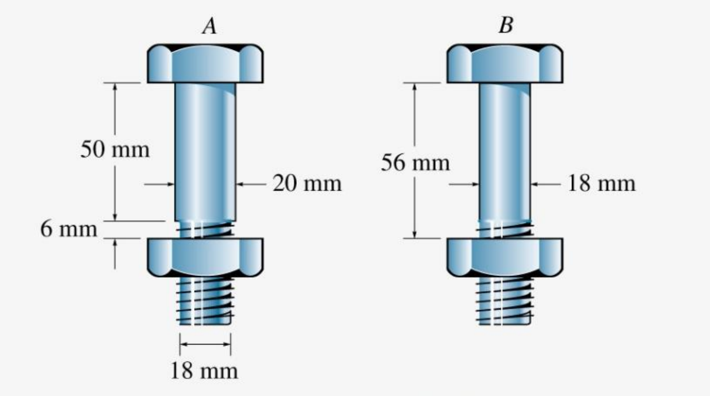
$$= 2707.8\ \text{N}\cdot\text{mm} = 2.708\ \text{N}\cdot\text{m} = 2.708\ \text{J}$$ ***Ans.***

Bolt B. Here the bolt is assumed to have a uniform diameter of 18 mm throughout its 56 mm length. Also, from the calculation above, it can support a maximum tension force of $P_{\max} = 78.89$ kN. Thus,

$$U_i = \frac{N^2L}{2AE} = \frac{(78.89\times10^3\ \text{N})^2(56\ \text{mm})}{2[\pi(18\ \text{mm}/2)^2][210(10^3)\ \text{N/mm}^2]}$$

$$= 3261.0\ \text{N}\cdot\text{mm} = 3.26\ \text{N}\cdot\text{m} = 3.26\ \text{J}$$ ***Ans.***

By comparison, bolt B can absorb 20% more elastic energy than bolt A, even though it has a smaller cross section along its shank.



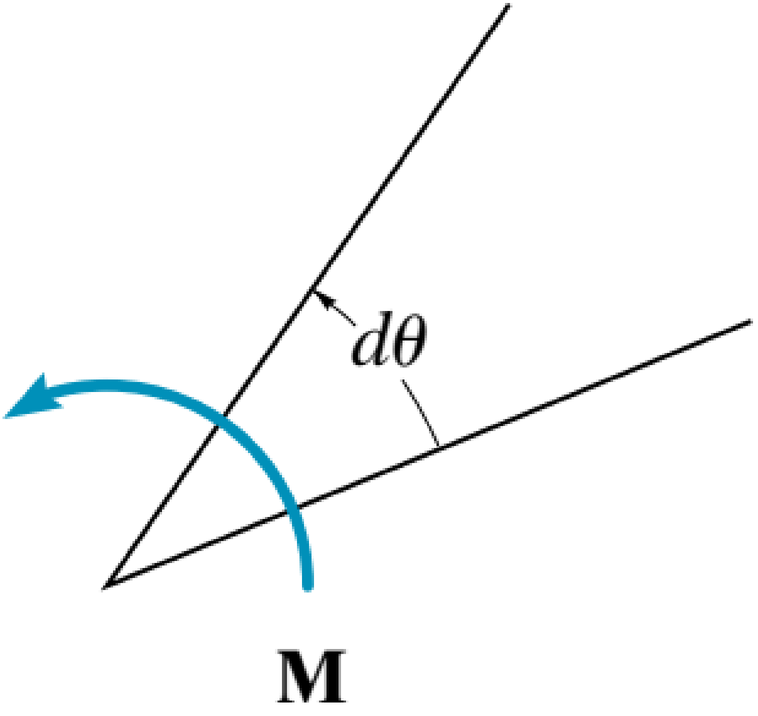
External Work and Strain Energy

- **External Work—Moment**. The work of a moment is defined by the product of the magnitude of the moment M and the angle $d\theta$ through which it rotates, that is, $d\boldsymbol{U_e} = M\,d\theta$. Hookes law holds. After integration

$$\boldsymbol{U_e} = \frac{\mathbf{1}}{\mathbf{2}} M\theta$$

- If the moment is already applied to the structure and other loadings further distort the structure by an amount θ', then M rotates θ', and the work is

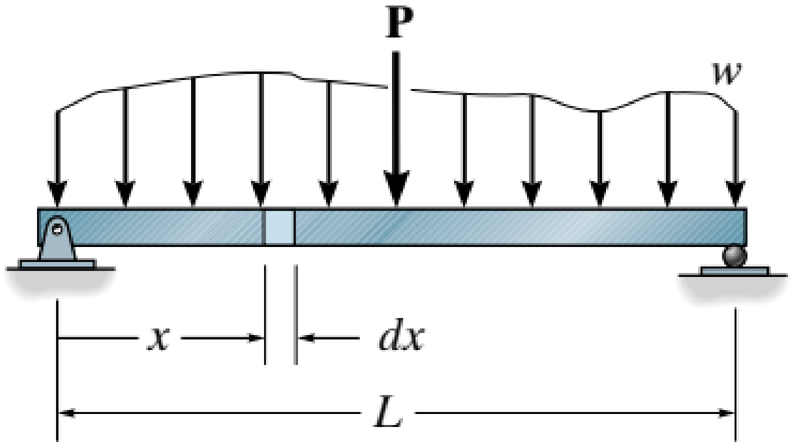
$$\boldsymbol{U'_e} = M\theta'$$



External Work and Strain Energy

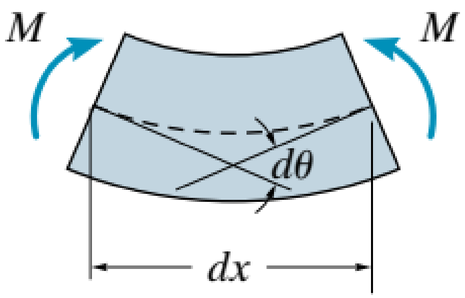
- **Strain Energy—Bending**. The loads P and w create an internal moment M in the beam at a section located a distance x from the left support. The rotation is given by $d\theta = M/EI\,dx$. Thus the strain energy is

$$dU_i = \frac{M^2\,dx}{2EI}$$

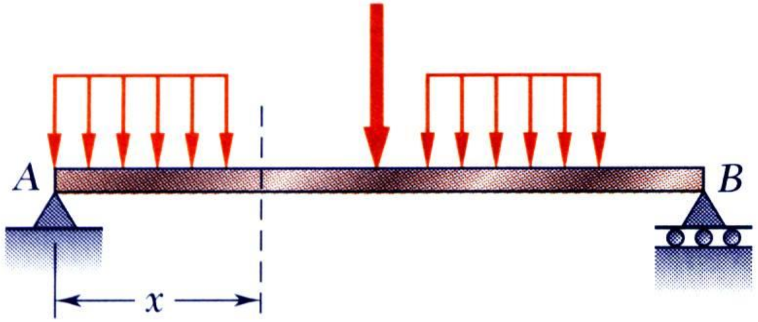


- Consequently, the strain energy, or work stored in the entire beam, is determined as

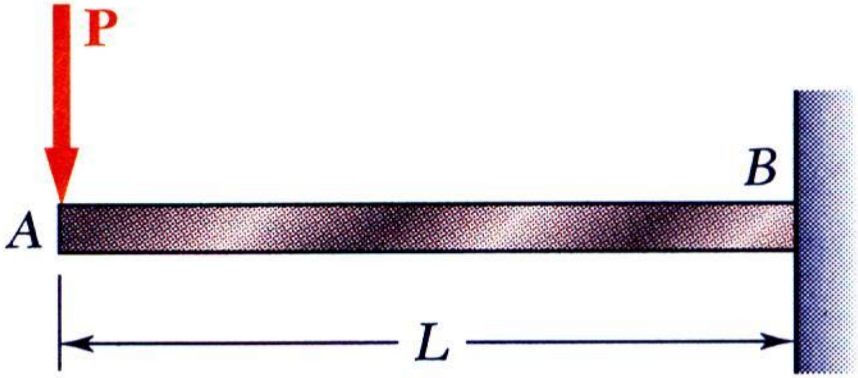
$$U_i = \int_0^L \frac{M^2\,dx}{2EI}$$



Strain Energy in Bending



$$\sigma_x = \frac{My}{I}$$



- For a beam subjected to a bending load,

$$U = \int \frac{\sigma_x^2}{2E} dV = \int \frac{M^2 y^2}{2EI^2} dV$$

- Setting $dV = dA\,dx$,

$$U = \int_0^L \int_A \frac{M^2 y^2}{2EI^2} dA\,dx = \int_0^L \frac{M^2}{2EI^2}\left(\int_A y^2 dA\right) dx$$

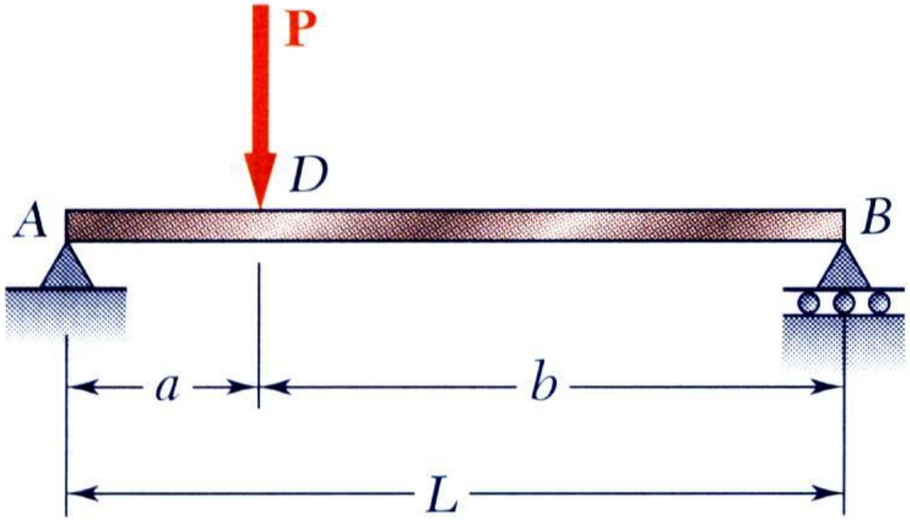
$$= \int_0^L \frac{M^2}{2EI} dx$$

- For an end-loaded cantilever beam,

$$M = -Px$$

$$U = \int_0^L \frac{P^2 x^2}{2EI} dx = \frac{P^2 L^3}{6EI}$$

Example 2



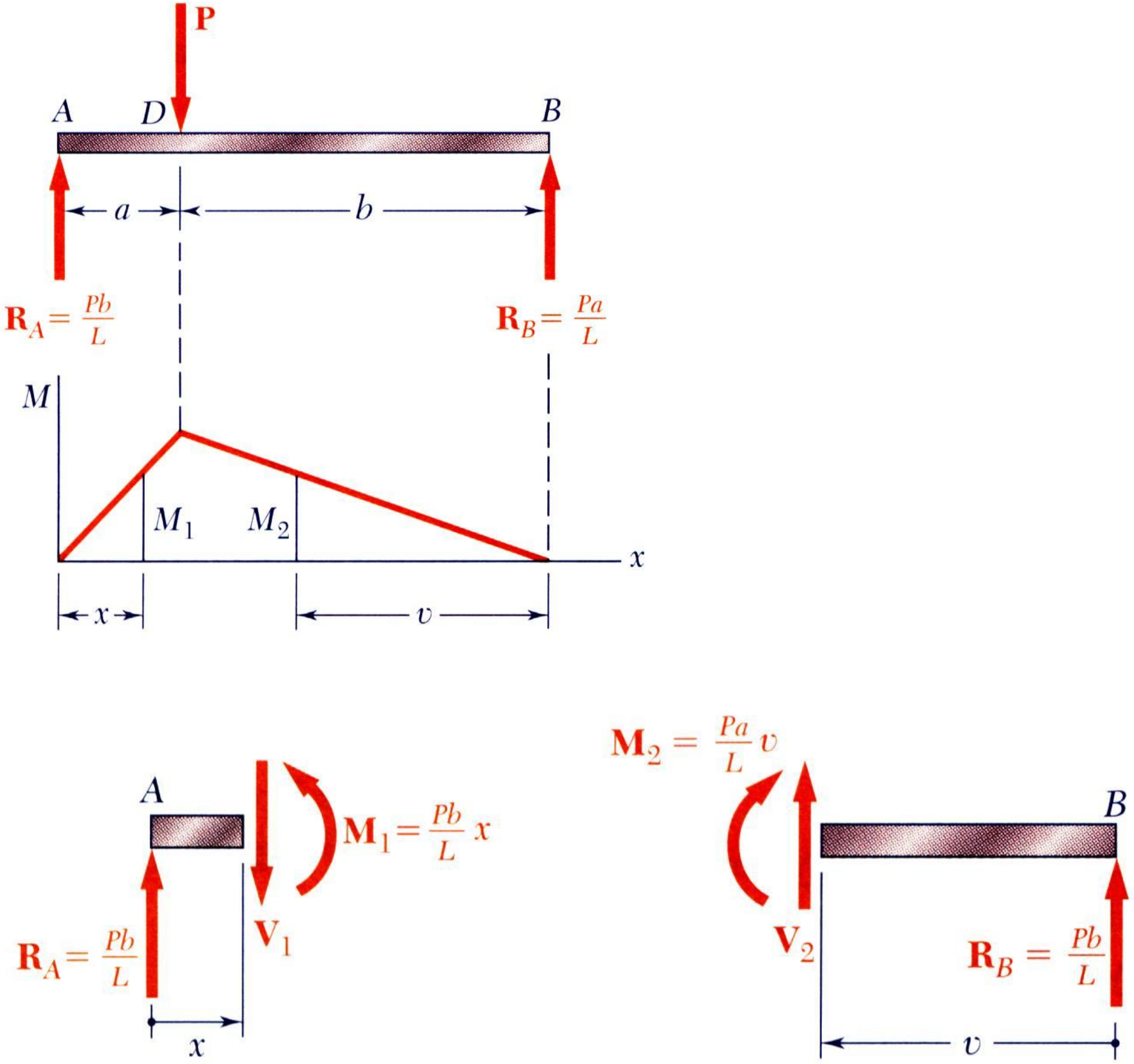
a) Taking into account only the normal stresses due to bending, determine the strain energy of the beam for the loading shown.

b) Evaluate the strain energy knowing that the beam is a W250x67, P = 160kN, L = 3.6m, a = 0.9m, b = 2.7m, and E = 200GPa.

SOLUTION:

- Determine the reactions at A and B from a free-body diagram of the complete beam.
- Develop a diagram of the bending moment distribution.
- Integrate over the volume of the beam to find the strain energy.
- Apply the particular given conditions to evaluate the strain energy.

Solution



SOLUTION:

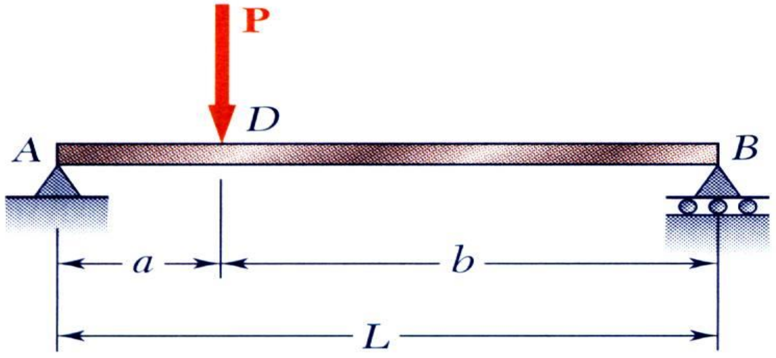
- Determine the reactions at *A* and *B* from a free-body diagram of the complete beam.

$$R_A = \frac{Pb}{L} \qquad R_B = \frac{Pa}{L}$$

- Develop a diagram of the bending moment distribution.

$$M_1 = \frac{Pb}{L} x \qquad M_2 = \frac{Pa}{L} v$$

Solution



Over the portion AD,

$$M_1 = \frac{Pb}{L}x$$

Over the portion BD,

$$M_2 = \frac{Pa}{L}v$$

$$\begin{aligned} &\mathrm{P} = 160\mathrm{kN} \qquad &&\mathrm{L} = 3.6\mathrm{m} \\ &\mathrm{a} = 0.9\mathrm{m} &&\mathrm{b} = 2.7\mathrm{m} \\ &\mathrm{E} = 200\mathrm{GPa} &&\mathrm{I} = 104\times 10^{6}\ \mathrm{mm}^{4} \end{aligned}$$

- Integrate over the volume of the beam to find the strain energy.

$$U = \int_0^a \frac{M_1^2}{2EI}dx + \int_0^b \frac{M_2^2}{2EI}dv$$

$$= \frac{1}{2EI}\int_0^a \left(\frac{Pb}{L}x\right)^2 dx + \frac{1}{2EI}\int_0^b \left(\frac{Pa}{L}v\right)^2 dv$$

$$= \frac{1}{2EI}\frac{P^2}{L^2}\left(\frac{b^2a^3}{3} + \frac{a^2b^3}{3}\right) = \frac{P^2a^2b^2}{6EIL^2}(a+b)$$

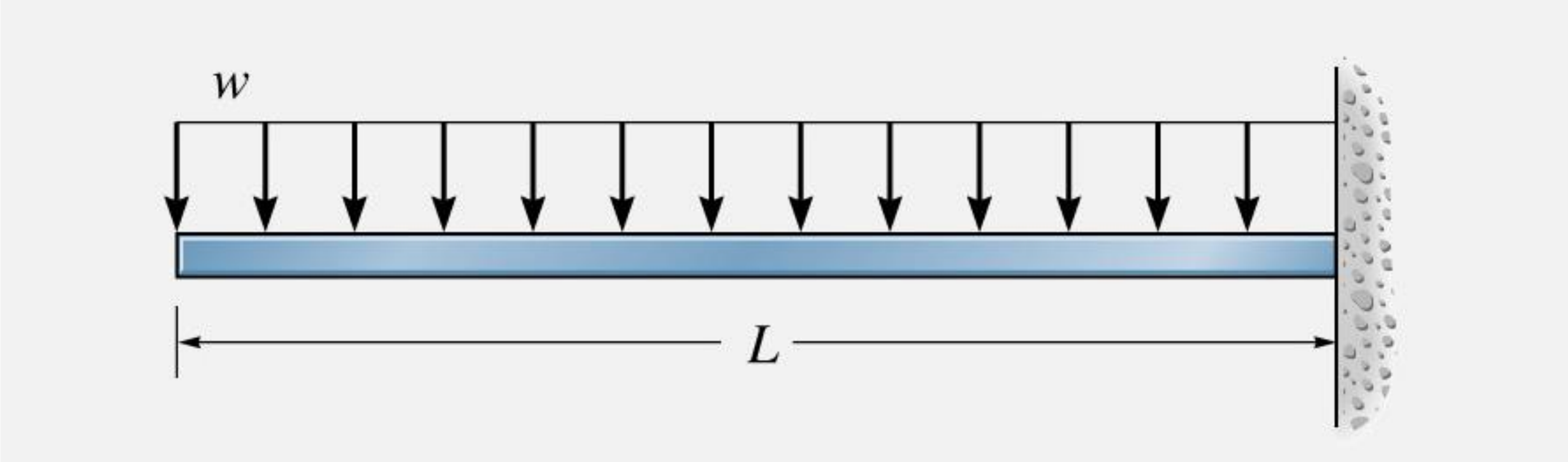
$$\boxed{U = \frac{P^2a^2b^2}{6EIL}}$$

$$\mathrm{U} = \frac{\left(160\times 10^3\ \mathrm{N}\right)^2 (0.9\mathrm{m})^2 (2.7\mathrm{m})^2}{6\left(200\times 10^9\ \mathrm{Pa}\right)\left(104\times 10^{-6}\ \mathrm{m}^4\right)(3.6\mathrm{m})}$$

$$\boxed{\mathrm{U} = 336\mathrm{Nm}}$$

Example 3

- Determine the elastic strain energy due to the bending of the cantilevered beam if the beam is subjected to a uniform distributed load w. EI is constant.



Solution:

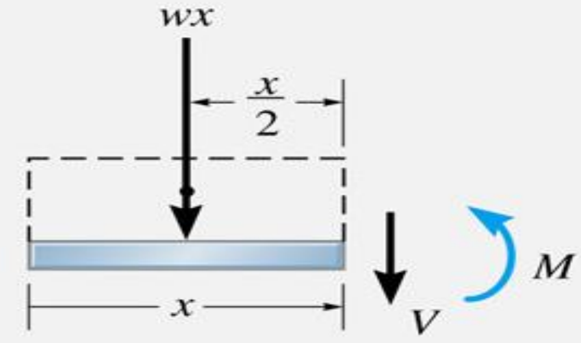
The internal moment in the beam is determined by establishing the x coordinate with origin at the left side. The left segment of the beam is shown in Fig. 14–10b. We have

$$\circlearrowleft + \Sigma M_{NA} = 0; \qquad M + wx\left(\frac{x}{2}\right) = 0$$

$$M = -w\left(\frac{x^2}{2}\right)$$

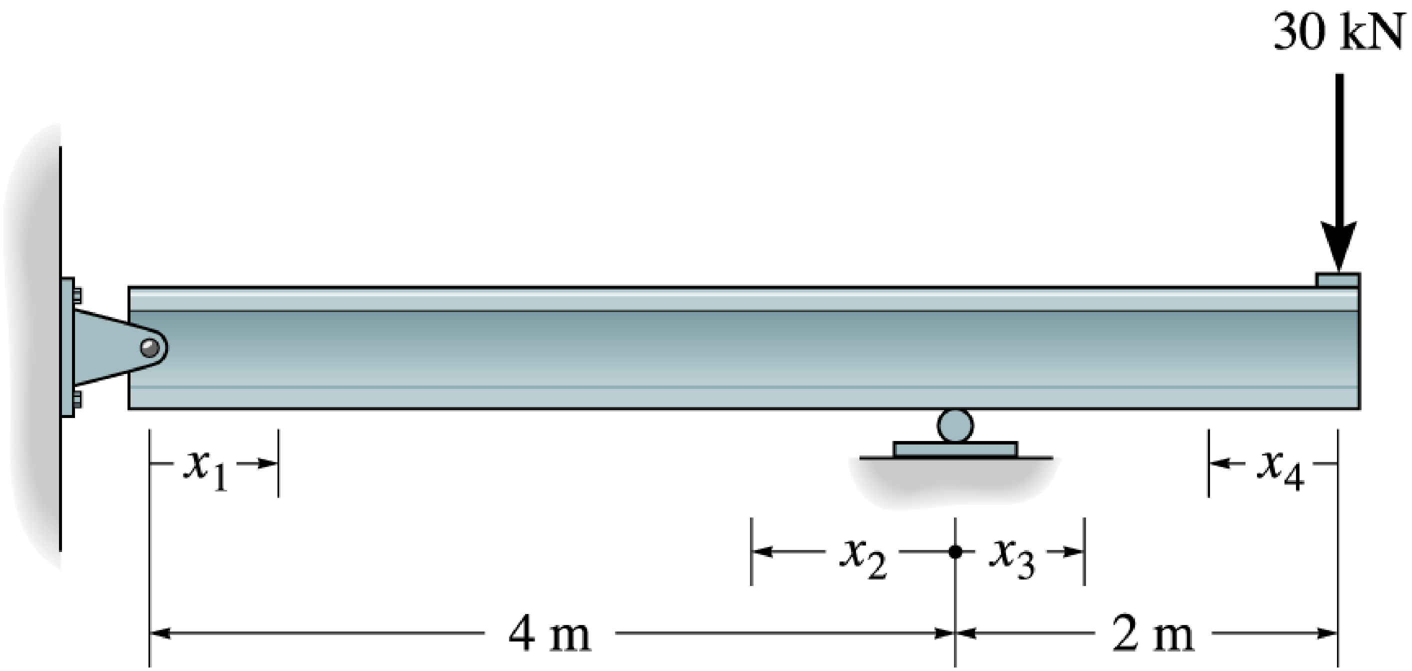
Applying Eq. 14–17 yields

$$U_i = \int_0^L \frac{M^2\,dx}{2EI} = \int_0^L \frac{[-w(x^2/2)]^2\,dx}{2EI} = \frac{w^2}{8EI}\int_0^L x^4\,dx$$



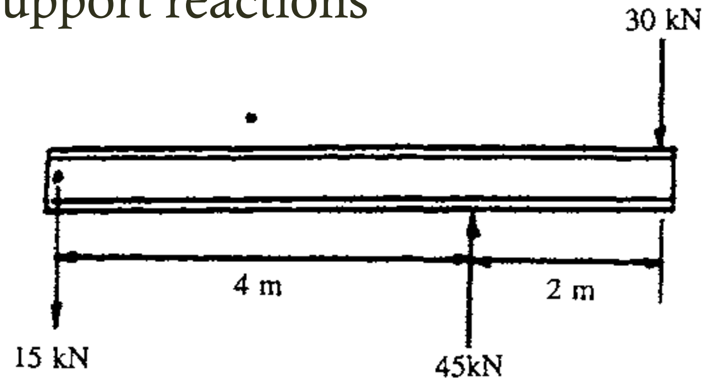
Example 4

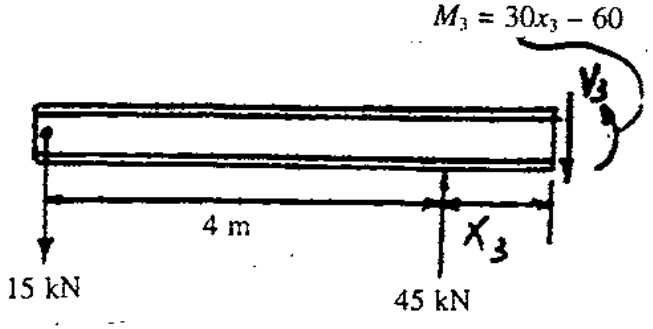
- Determine the bending strain energy in the A-36 structural steel $W250 \times 18$ beam. Obtain the answer using the cooordinaes $(a)\ x_1\ and\ x_4$ and $(b)\ x_2\ and\ x_3$. $E = 210\ GPa$.



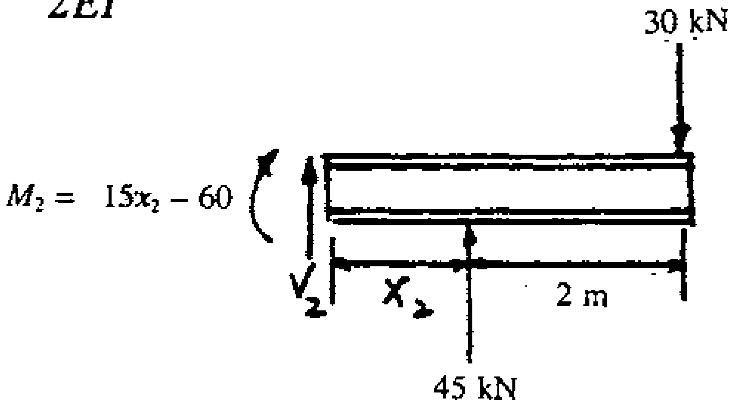
Solution

- Finding the support reactions





2EI



- Finding the internal moment

$$M_3 = 30x_3 - 60$$

$$M_2 = 45x_2 - 30(x_2 + 2) = 15x_2 - 60$$

$$U_i = \int_0^4 \frac{(15x_2 - 60)^2\,dx_2}{2EI} + \int_0^2 \frac{(30x_3 - 60)^2\,dx_3}{2EI}$$

$$= \int_0^4 \frac{(225x_2^2 - 1800x_2 + 3600)dx_2}{2EI} + \int_0^2 \frac{(900x_3^2 - 3600x_3 + 3600)dx_3}{2EI}$$

Solution (Contd.)

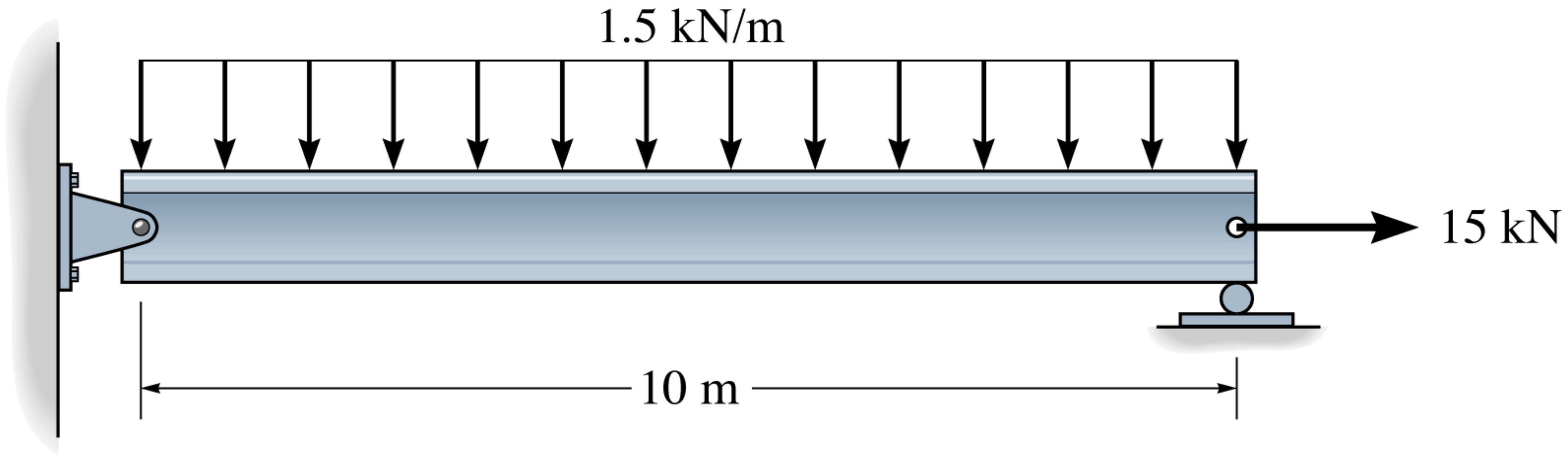
$$= \frac{1}{2EI}\left[225\left(\frac{4^3}{3}\right) - 1800\left(\frac{4^2}{2}\right) + 3600(4)\right.$$

$$\left. + 900\left(\frac{2^3}{3}\right) - 3600\left(\frac{2^2}{2}\right) + 3600(2)\right]$$

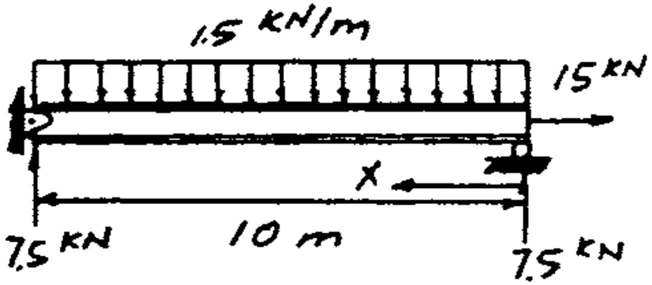
$$= \frac{3600\ \text{kN}^2 \cdot \text{m}^3}{210(10^6)(22.5)(10^{-6})} = 762\ \text{J} \qquad \textbf{Ans}$$

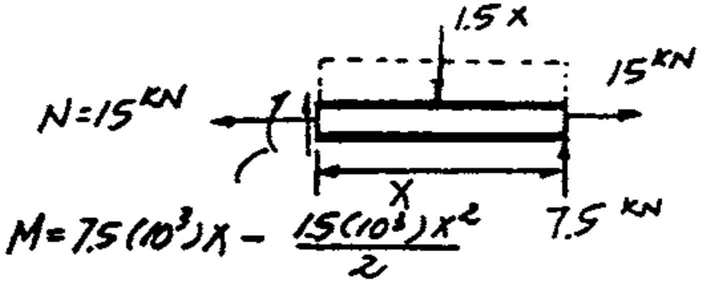
Example 5

- Determine the total axial and bending strain energy in the A-36 structural steel beam. $A = 2300\,mm^2$, $I = 9.5 \times 10^6\,mm^4$, $E = 210\,GPa$.



Solution





Bending :

$$(U_b)_i = \int_0^L \frac{M^2 dx}{2EI} = \frac{1}{2EI}\int_0^{10} [(7.5)(10^3)x - 0.75(10^3)x^2]^2 dx$$

$$= \frac{1}{2EI}\int_0^{10} [56.25(10^6)x^2 + 562.5(10^3)x^4 - 11.25(10^6)x^3]\,dx$$

$$(U_b)_i = \frac{0.9375(10^9)}{200(10^9)(9.5)(10^{-6})} = 493.4210 \text{ J}$$

Axial load :

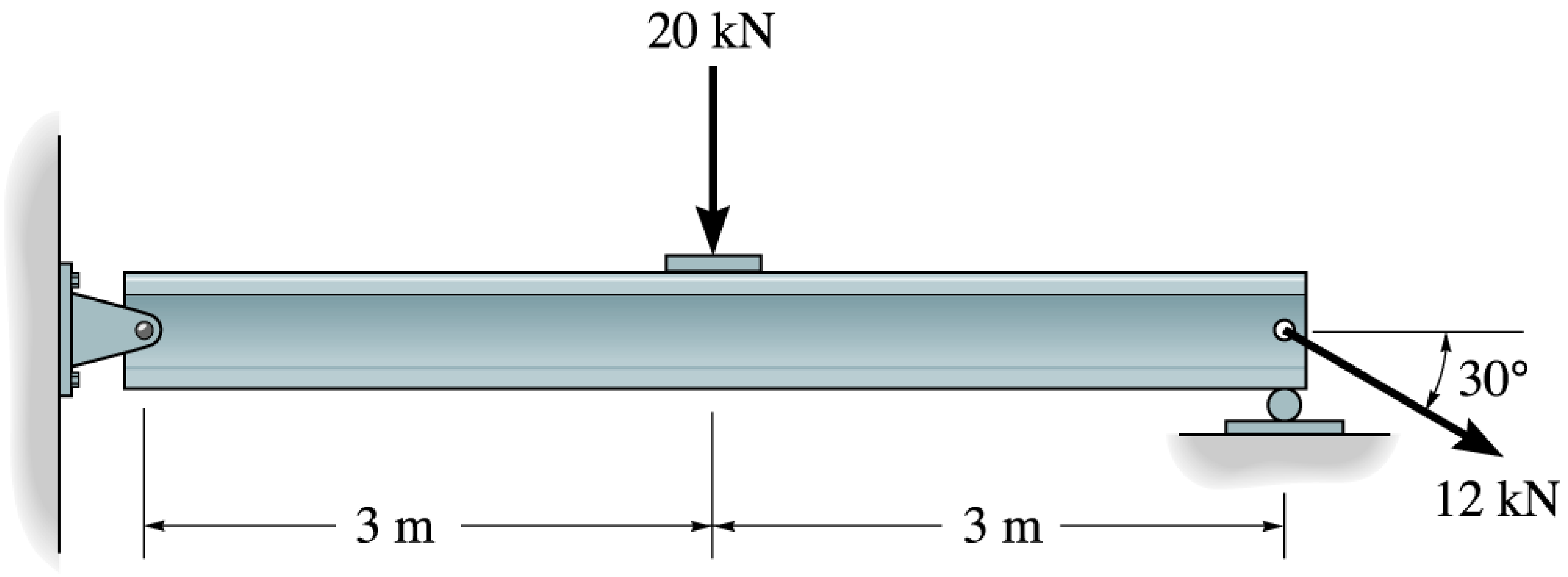
$$(U_a)_i = \int_0^L \frac{N^2 dx}{2EA} = \frac{N^2 L}{2EA}$$

$$(U_a)_i = \frac{((15)(10^3))^2(10)}{2(200)(10^9)(2.3)(10^{-3})} = 2.4456 \text{ J}$$

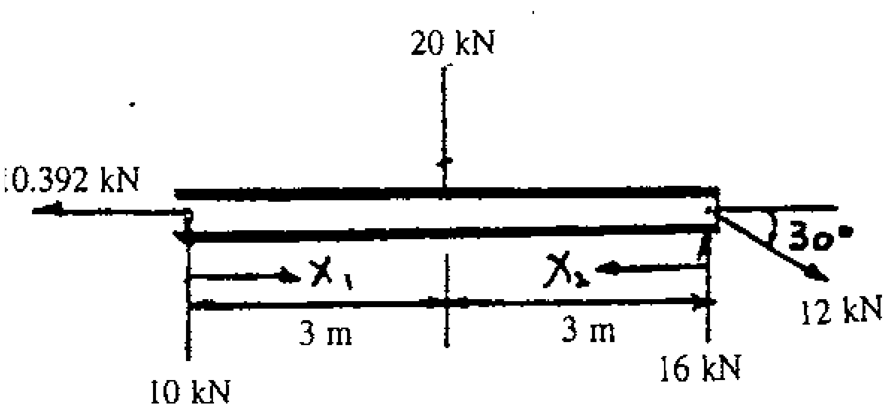
$$U_i = (U_a)_i + (U_b)_i = 2.4456 + 493.4210 = 496 \text{ J} \qquad \text{Ans}$$

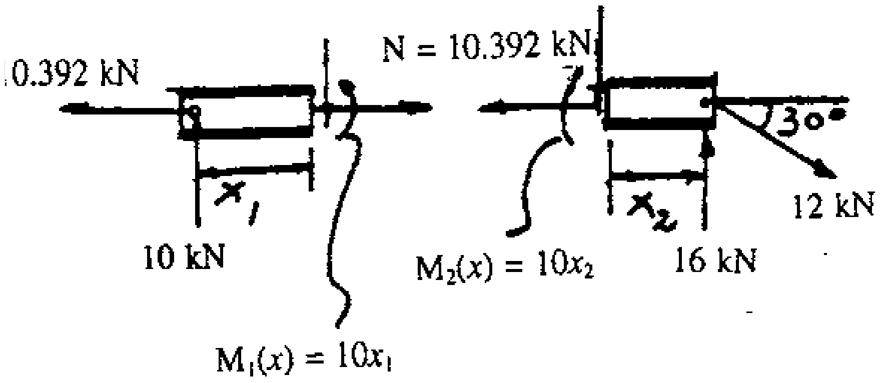
Example 6

Determine the total axial and bending strain energy in the A-36 structural steel W200 × 86 beam. $E = 210$ GPa.



Solution





Axial load :

$$(U_a)_i = \int_0^L \frac{N^2 L}{2AE} = \frac{N^2 L}{2AE}$$

$$= \frac{[10.392(10^3)]^2(6)}{2(11\,000)(10^{-6})(210)(10^9)} = 0.140 \text{ N}\cdot\text{m} = 0.14 \text{ J}$$

Bending :

$$(U_b)_i = \int_0^L \frac{M^2 dx}{2EI} = \frac{2}{2EI}\int_0^3 [10(10^3)x]^2\, dx$$

$$= \frac{900(10^6)}{EI} = \frac{900(10^6)}{210(10^9)(94.7)(10^{-6})} = 45.26 \text{ N}\cdot\text{m} = 45.26 \text{ J}$$

Total strain energy :

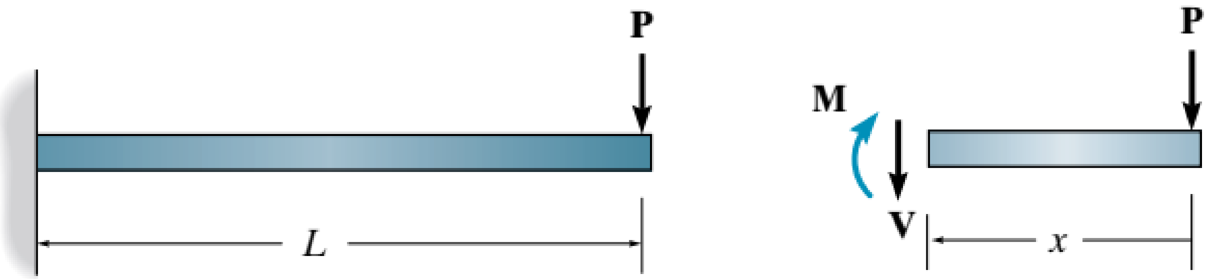
$$U_i = (U_a)_i + (U_b)_i$$

$$= 0.14 + 45.26$$

$$= 45.4 \text{ J} \qquad \textbf{Ans}$$

Principle of Virtual Work

- We can use the principle of work and energy can be applied to determine the displacement at a point on a structure.
- For a cantilever, we determine the displacement at the end as



$$M = -Px$$

$$U_i = \int_0^L \frac{M^2\,dx}{2EI} = \int_0^L \frac{(-Px)^2\,dx}{2EI} = \frac{1}{6}\frac{P^2L^3}{EI}$$

$$U_e = U_i$$

$$\frac{1}{2}P\Delta = \frac{1}{6}\frac{P^2L^3}{EI}$$

$$\Delta = \frac{PL^3}{3EI}$$

- It will be noted that only one load may be applied to the structure, since if more than one load were applied, there would be an unknown displacement under each load, and yet it is possible to write only one "work" equation for the beam.
- Furthermore, only the displacement under the force can be obtained, since the external work depends upon both the force and its corresponding displacement.

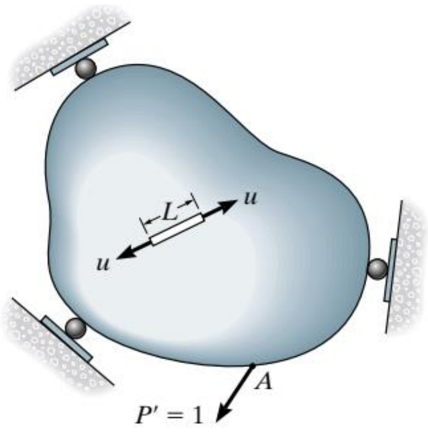
Principle of Virtual Work

- The external and internal displacements must be related by the compatibility of the displacements.
- The principle was developed by John Bernoulli in 1717 and is sometimes referred to as the **unit-load method**.
- Applying a series of external loads P to it, it will cause internal loads u at points throughout the structure. It is necessary that the external and internal loads be related by the equations of equilibrium.

$$\underbrace{\Sigma P\Delta}_{\text{Work of External Loads}} = \underbrace{\Sigma u\delta}_{\text{Work of Internal Loads}}$$

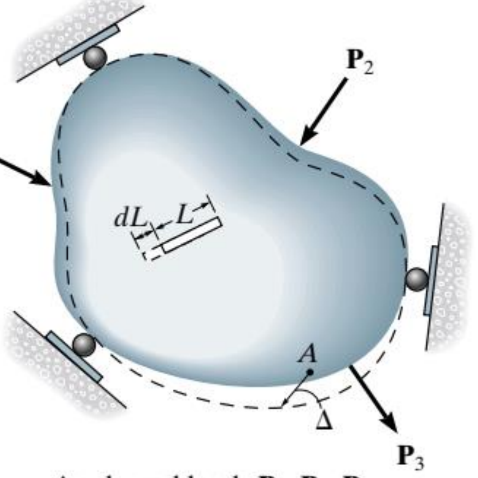
Principle of Virtual Work

- Suppose it is necessary to determine the displacement Δ of point A on the body caused by the "real loads" $P_1,\ P_2,\ and\ P_3$.
- Since no external load acts on the body at A and in the direction of Δ, the displacement Δ can be determined by first placing on the body a "virtual" load such that this force $P' = 1$ acts in the same direction as Δ.
- The unit load (P') does not exist in reality, create an internal virtual load u in a representative element or fiber of the body.



Apply virtual load $P' = 1$

(a)



Apply real loads $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$

Principle of Virtual Work

- Point A will be displaced an amount Δ, causing the element to deform an amount dL.
  - As a result, the external virtual force P' and internal virtual load u “ride along” by Δ and dL, respectively, therefore perform external virtual work of $1 \times \Delta$ on the body and internal virtual work of $\sum u \times \boldsymbol{dL}$ on the element.
  - The external virtual work is equal to the internal virtual work done

$$1 \times \Delta = \sum u \times \boldsymbol{dL}$$

External Work

Internal virtual Work

where

$P' = 1 =$ external virtual unit load acting in the direction of Δ.

$u =$ internal virtual load acting on the element in the direction of dL.

$\Delta =$ external displacement caused by the real loads.

$dL =$ internal deformation of the element caused by the real loads.

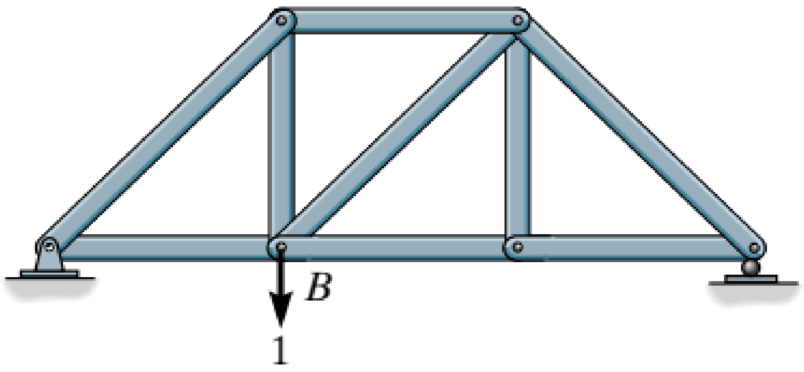
Method of Virtual Work: Trusses

- Consider the vertical displacement Δ of joint B of the truss
- If the applied loadings P_1 and P_2 cause a linear elastic material response, then this element deforms an amount $\Delta L = NL/AE$, where N is the normal or axial force in the member, caused by the loads.
- The virtual-work equation for the truss.

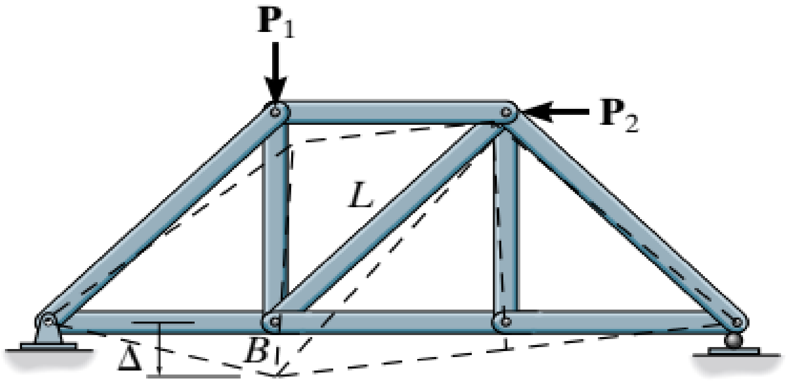
$$1 \times \Delta = \sum \frac{nNL}{AE}$$

- 1 = external virtual unit load acting on the truss joint in the stated direction of Δ.
  n = internal virtual normal force in a truss member caused by the external virtual unit load.
  Δ = external joint displacement caused by the real loads on the truss.
  N = internal normal force in a truss member caused by the real loads.
  L = length of a member.
  A = cross-sectional area of a member.
  E = modulus of elasticity of a member

Due to external loading



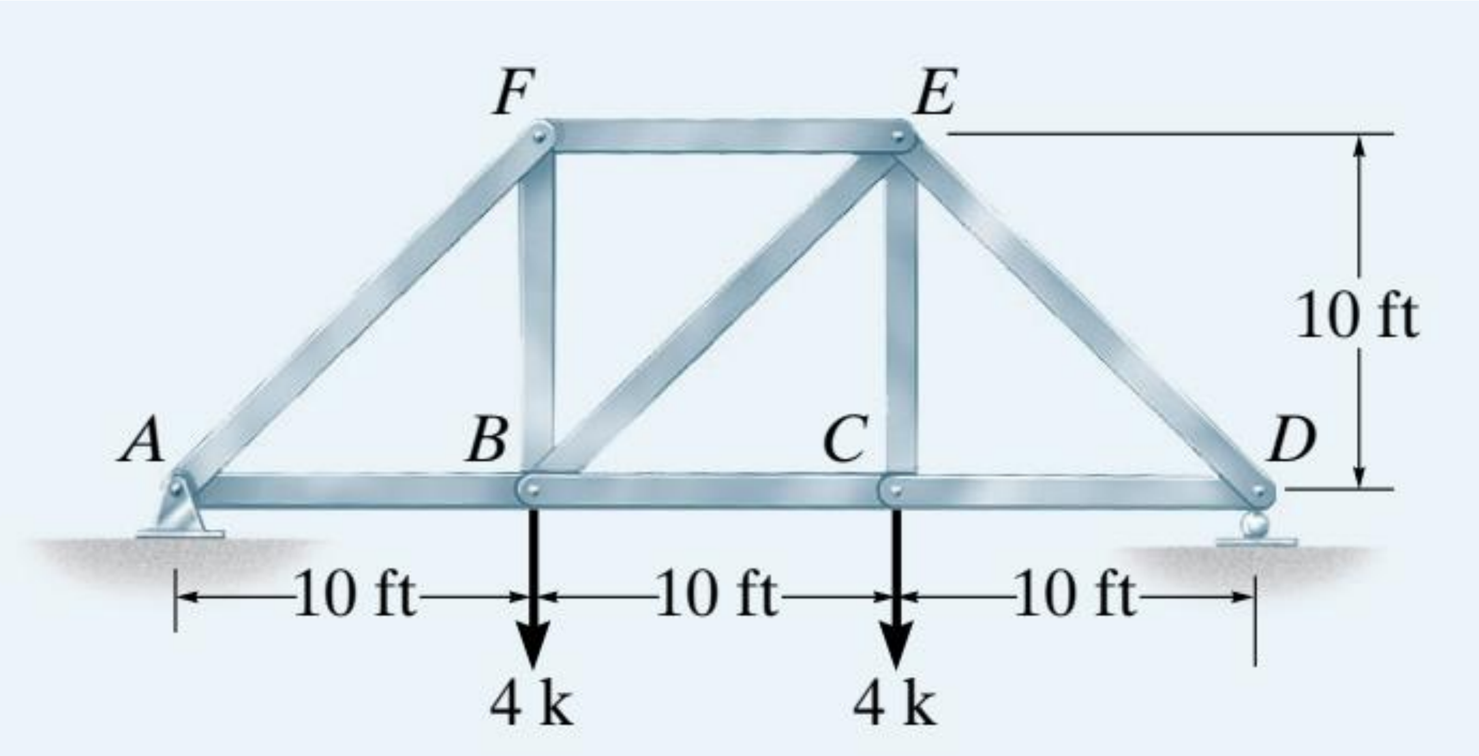
Apply virtual unit load to B



Apply real loads $\mathbf{P}_1$, $\mathbf{P}_2$

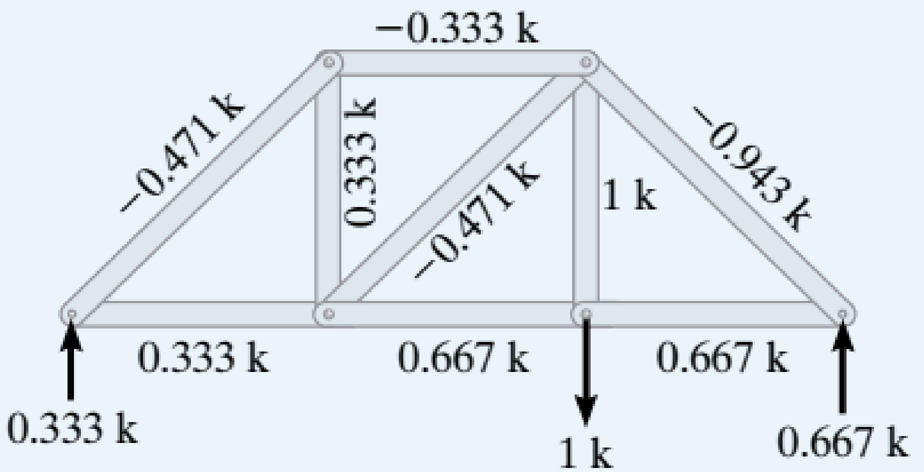
Example 7

- Determine the vertical displacement of joint C of the steel truss shown in the figure. The cross-sectional area of each member is $A = 0.5\ in^2$ and $E = 29(10^3)\ ksi$.



Solution 7

Member	n (k)	N (k)	L (ft)	nNL ($k^2 \cdot ft$)
AB	0.333	4	10	13.33
BC	0.667	4	10	26.67
CD	0.667	4	10	26.67
DE	−0.943	−5.66	14.14	75.42
FE	−0.333	−4	10	13.33
EB	−0.471	0	14.14	0
BF	0.333	4	10	13.33
AF	−0.471	−5.66	14.14	37.71
CE	1	4	10	40
				Σ246.47

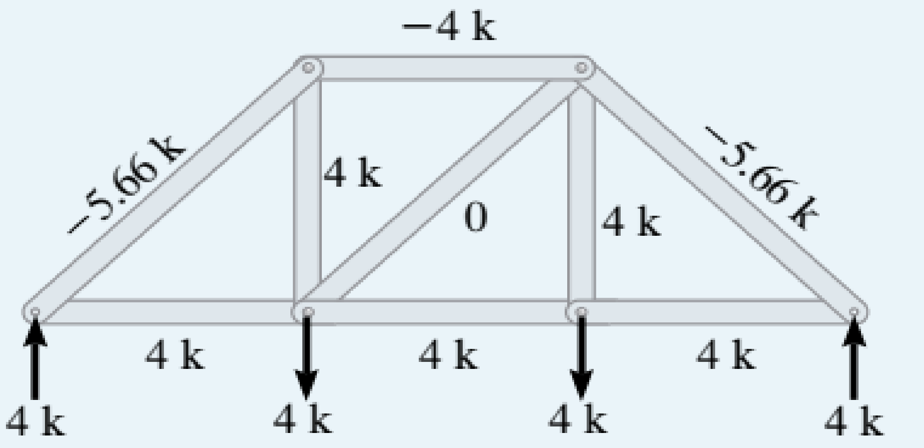


virtual forces **n**

$$1\ \mathrm{k} \cdot \Delta_{C_v} = \sum \frac{nNL}{AE} = \frac{246.47\ \mathrm{k}^2 \cdot \mathrm{ft}}{AE}$$

$$1\ \mathrm{k} \cdot \Delta_{C_v} = \frac{(246.47\ \mathrm{k}^2 \cdot \mathrm{ft})(12\ \mathrm{in./ft})}{(0.5\ \mathrm{in}^2)(29(10^3)\ \mathrm{k/in}^2)}$$

$$\Delta_{C_v} = 0.204\ \mathrm{in.}$$



real forces **N**

Method of Virtual Work: Trusses

Due to Temperature

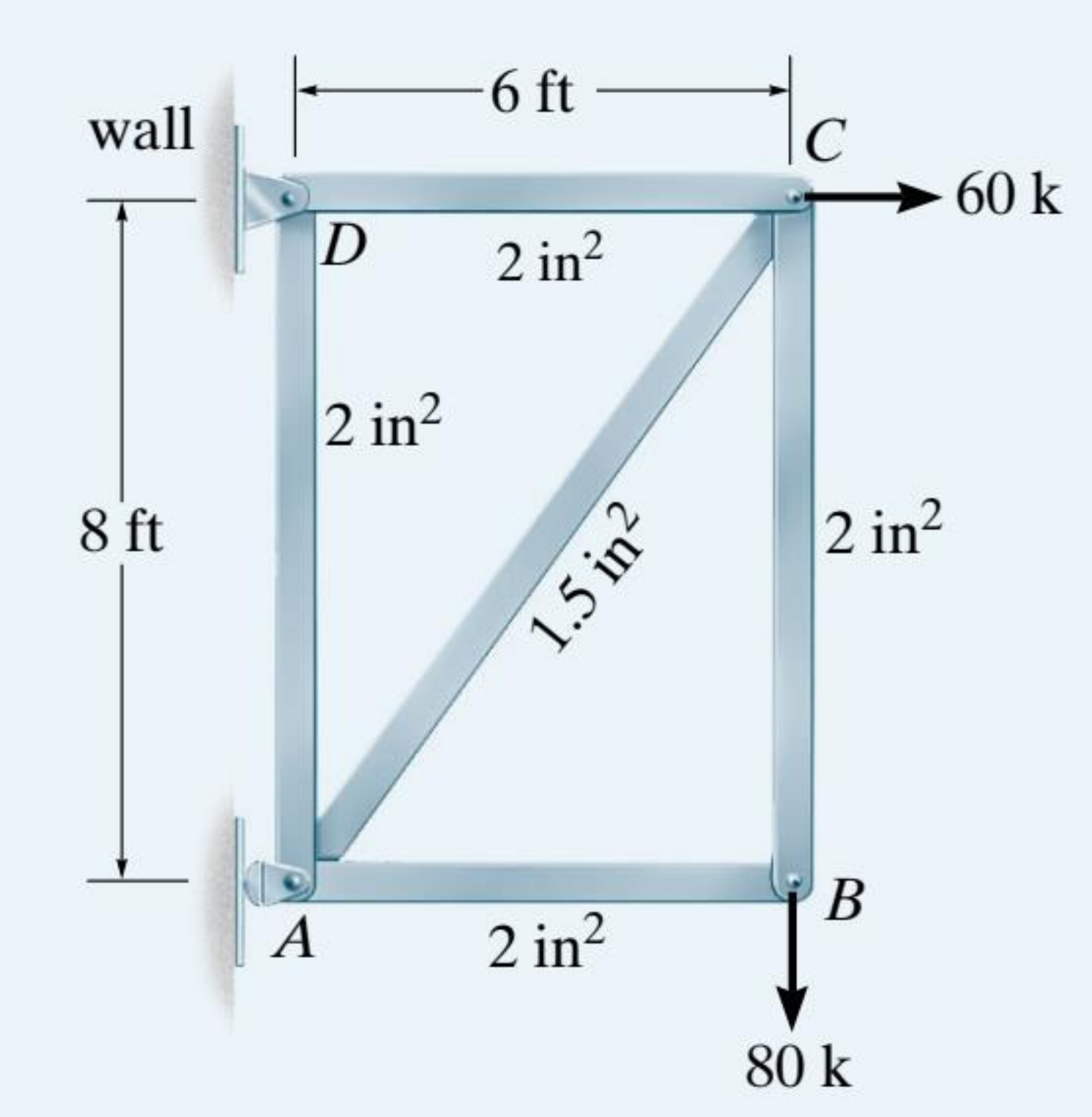
- In some cases, truss members may change their length due to temperature.
- If α is the coefficient of thermal expansion for a member and ΔT is the change in its temperature, the change in length of a member is $\Delta L = \boldsymbol{\alpha}\Delta T\, L$.
- The virtual-work equation for the truss due to temperature is.

$$\mathbf{1} \times \Delta = \sum \boldsymbol{n\alpha L\Delta T}$$

- 1 = external virtual unit load acting on the truss joint in the stated direction of Δ.
  n = internal virtual normal force in a truss member caused by the external virtual unit load.
  Δ = external joint displacement caused by the real loads on the truss.
  α = coefficient of thermal expansion of member.
  ΔT = change in temperature of member.
  L = length of member.

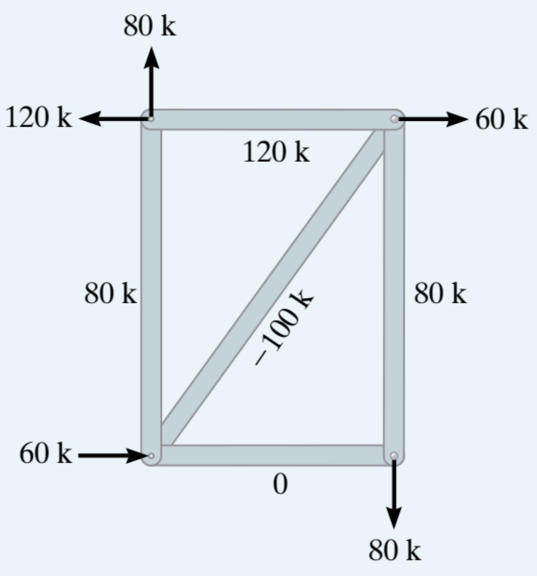
Example 8

- Determine the vertical displacement of joint C of the steel truss shown in the figure. Due to radiant heating from the wall, member AD is subjected to an *increase* in temperature of $\Delta T = +120°F$. Take $\alpha = 0.6(10^{-5})/°F$ and $E = 29(10^3)\ ksi$. The cross-sectional area of each member is indicated in the figure.

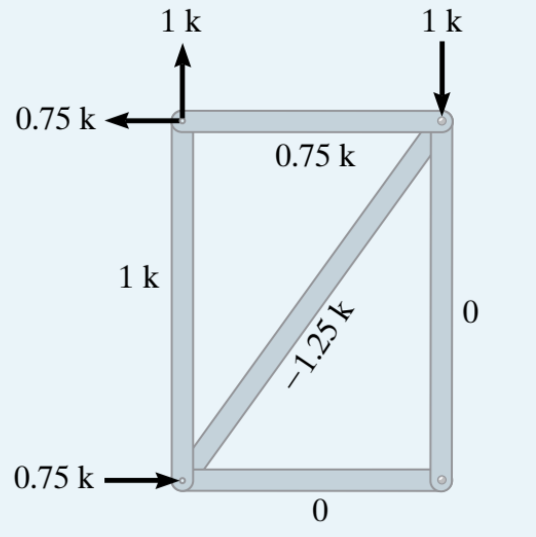


Solution 8

- Since the n forces in members AB and BC are zero, the N forces in these members do not have to be computed. Why?



real forces **N**



virtual forces **n**

$$1 \cdot \Delta_{C_v} = \sum \frac{nNL}{AE} + \Sigma n\alpha\, \Delta T\, L$$

$$= \frac{(0.75)(120)(6)(12)}{2\left[29(10^3)\right]} + \frac{(1)(80)(8)(12)}{2\left[29(10^3)\right]}$$

$$+ \frac{(-1.25)(-100)(10)(12)}{1.5\left[29(10^3)\right]} + (1)\left[0.6(10^{-5})\right](120)(8)(12)$$

$$\Delta_{C_v} = 0.658 \text{ in.} \qquad \textit{Ans.}$$

Method of Virtual Work: Trusses

- In some cases truss members must be made slightly longer or shorter in order to give the truss a camber.
- In some cases, errors in fabricating the lengths of the members of a truss may occur
- If a truss member is shorter or longer than intended

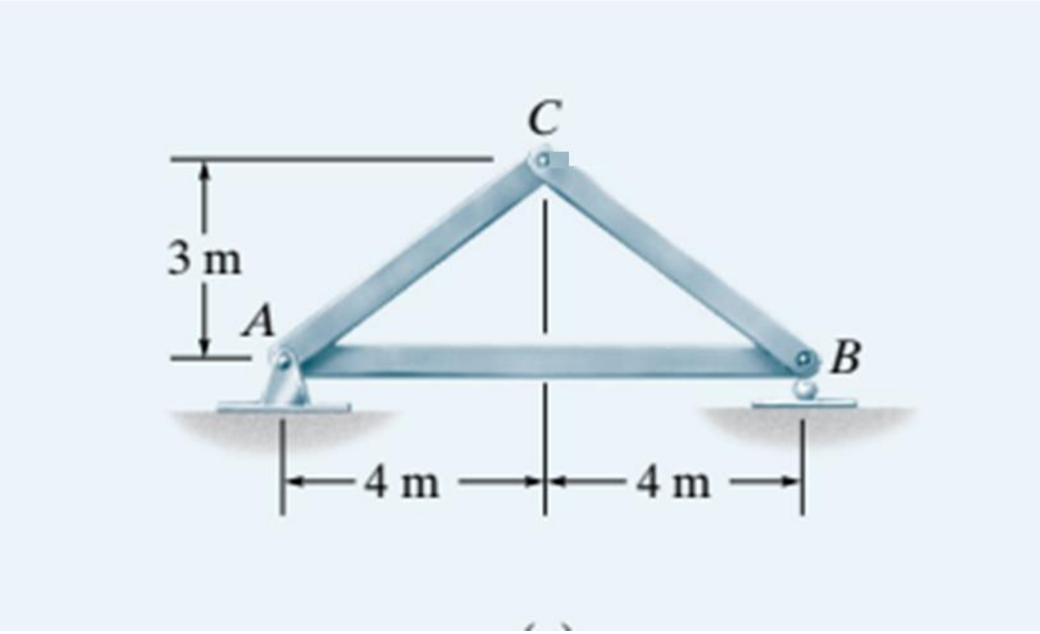
Due to Fabrication Errors

$$1 \times \Delta = \sum n\Delta L$$

- 1 = external virtual unit load acting on the truss joint in the stated direction of Δ.
  n = internal virtual normal force in a truss member caused by the external virtual unit load.
  Δ = external joint displacement caused by the real loads on the truss.
  ΔL = difference in length of the member from its intended size as caused by a fabrication error.

Example 9

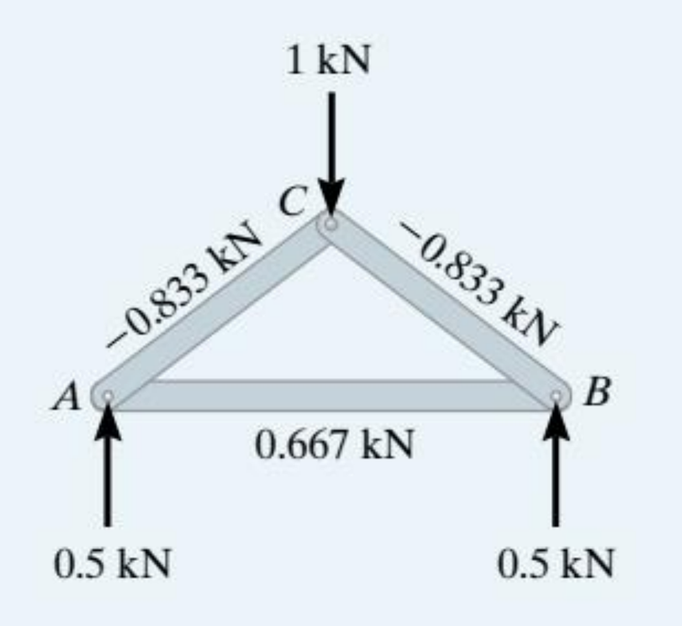
- The cross-sectional area of each member of the truss shown in the figure is $A = 400\ mm^2$ and $E = 200\ GPa$. If no loads act on the truss, what would be the vertical displacement of joint C if member AB were 5 mm too short?



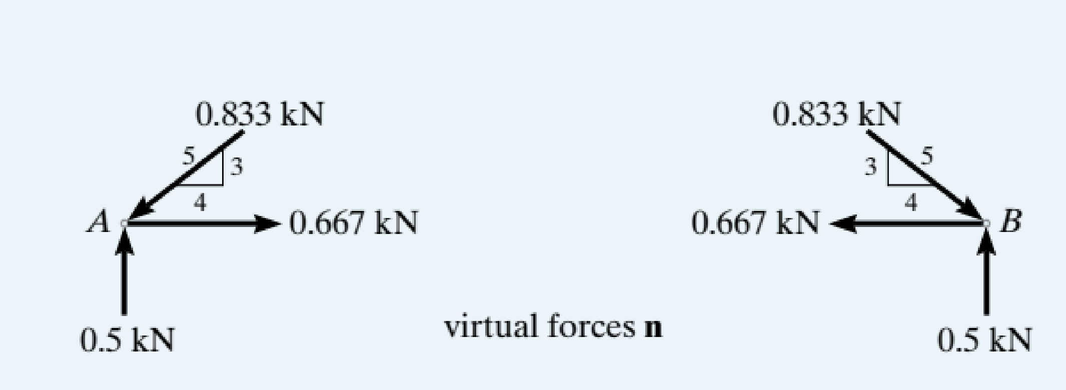
Solution 9

- Note that $\Delta L = 0.005\ m$

$$
\begin{aligned}
1 \cdot \Delta &= \Sigma n\, \Delta L \\
1\ \text{kN} \cdot \Delta_{C_v} &= (0.667\ \text{kN})(-0.005\ \text{m}) \\
\Delta_{C_v} &= -0.00333\ \text{m} = -3.33\ \text{mm} \qquad \textit{Ans.}
\end{aligned}
$$



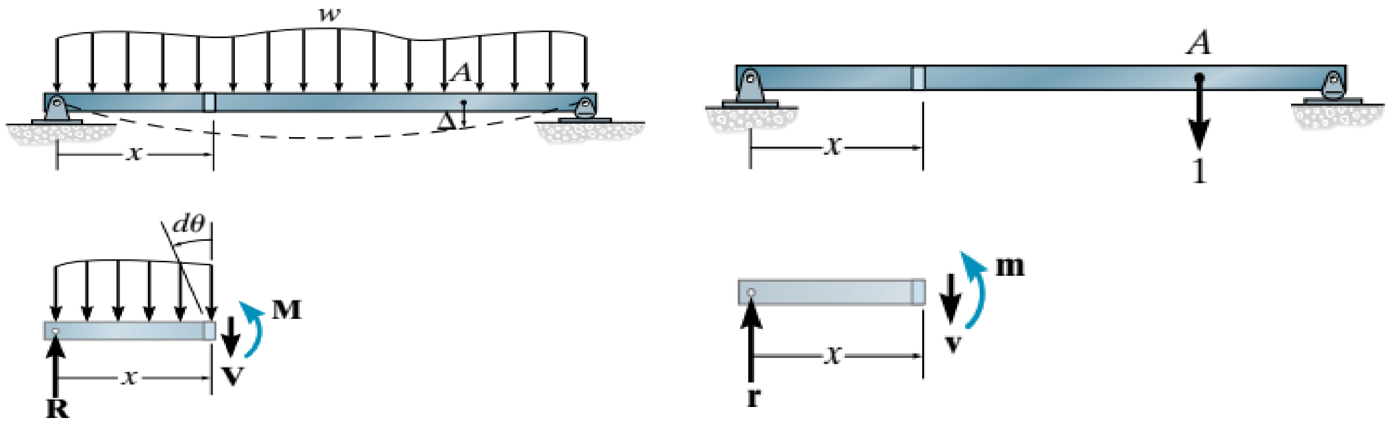
- The negative sign indicates joint C is displaced upward, opposite to the 1-kN vertical load
- If the deflection is as a result of both external loading and errors, the sum of both is necessary



Method of Virtual Work: Beams and Frames

- **Deflection Interms of Displacement**: Note: Strains due to bending are the primary cause of beam or frame deflections,

- Consider the beam shown in the figure. Here the displacement Δ of point A is to be determined. To compute Δ a virtual unit load acting in the direction of Δ is placed on the beam at *A*, and the internal virtual moment *m* is determined by the method of sections at an arbitrary location *x* from the left support



Apply real load *w*

Apply virtual unit load to point *A*

$$d\theta = m(M/EI)\,dx$$

$$1 \times \Delta = \int_0^L \frac{mM}{EI}\,dx$$

where

1 = external virtual unit load acting on the beam or frame in the direction of Δ.

m = internal virtual moment in the beam or frame, expressed as a function of x and caused by the external virtual unit load.

Δ = external displacement of the point caused by the real loads acting on the beam or frame.

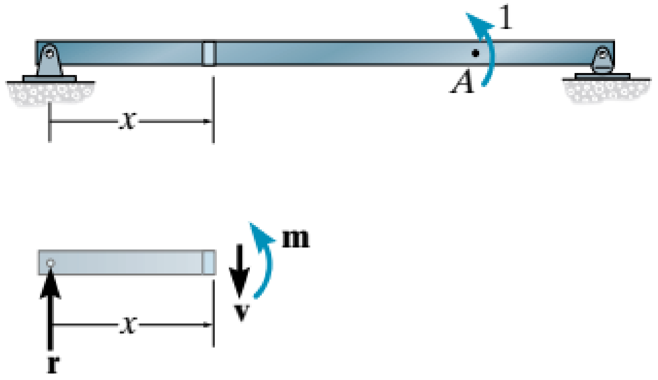
M = internal moment in the beam or frame, expressed as a function of x and caused by the real loads.

E = modulus of elasticity of the material.

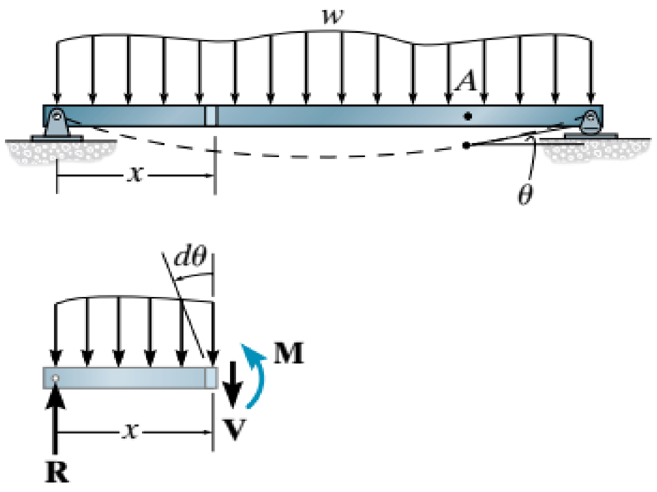
I = moment of inertia of cross-sectional area, computed about the neutral axis.

Method of Virtual Work: Beams and Frames

- **Deflection Interms of Slope Angle**: Note: Strains due to bending are the primary cause of beam or frame deflections,



Apply virtual unit couple moment to point A



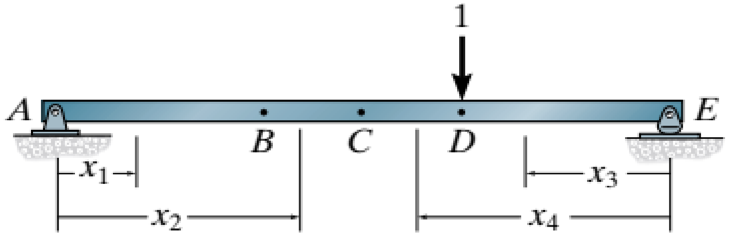
Apply real load w

- If the tangent rotation or slope angle θ at a point A on the beam's elastic curve is to be determined, a unit couple moment is first applied at the point.
- The corresponding internal moments m_θ have to be determined. Since the work of the unit couple is $1 \times \boldsymbol{\theta}$, then it follows that

$$\mathbf{1} \times \boldsymbol{\theta} = \int_0^L \frac{m_\theta \boldsymbol{M}}{\boldsymbol{EI}} \boldsymbol{dx}$$

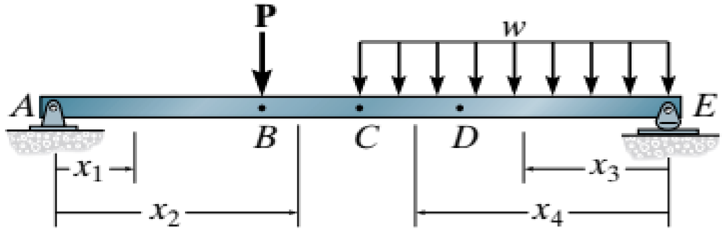
Method of Virtual Work: Beams and Frames

- If concentrated forces or couple moments act on the beam or the distributed load is discontinuous, a single integration cannot be performed across the beam's entire length.
- Instead, separate x coordinates will have to be chosen within regions that have no discontinuity of loading. Also, it is not necessary that each x have the same origin; however, the x selected for determining the real moment M in a particular region must be the same x as that selected for determining the virtual moment m or m_θ within the same region.
- Each x coordinate should be selected so that both M and m (or m_θ) can be easily formulated.



Apply virtual unit load

(a)



Apply real loads

(b)

Method of Virtual Work: Beams and Frames

Procedure for Analysis

The following procedure may be used to determine the displacement and/or the slope at a point on the elastic curve of a beam or frame using the method of virtual work.

Virtual Moments m or m

- Place a *unit load* on the beam or frame at the point and in the direction of the desired *displacement*.
- If the *slope* is to be determined, place a *unit couple moment* at the point.
- Establish appropriate x coordinates that are valid within regions of the beam or frame where there is no discontinuity of real or virtual load.
- With the virtual load in place, and all the real loads *removed* from the beam or frame, calculate the internal moment m or m_θ as a function of each x coordinate.
- Assume m or m_θ acts in the conventional positive direction as indicated in Fig. 4–1.

Real Moments

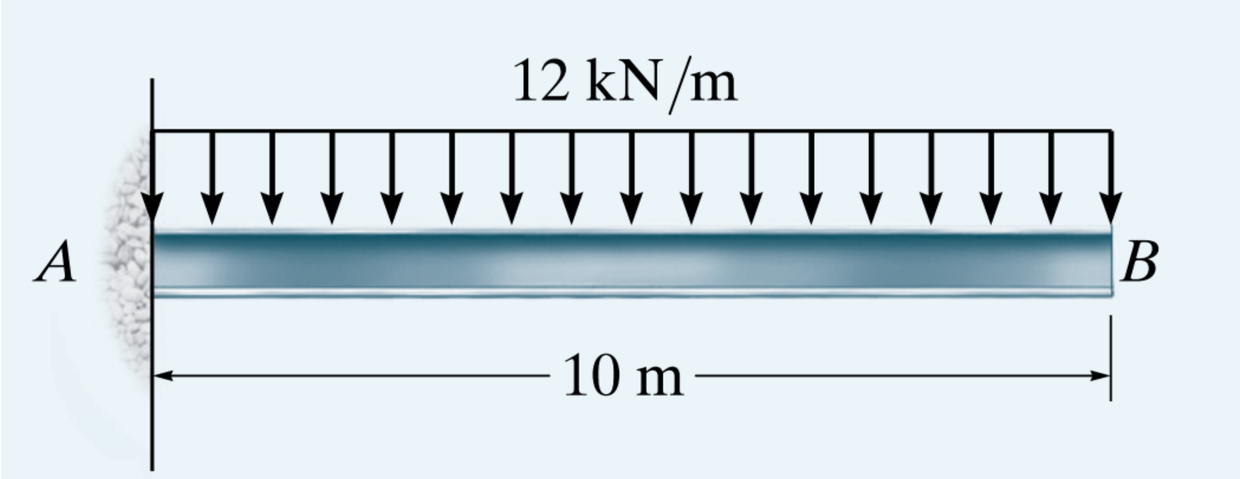
- Using the *same* x coordinates as those established for m or m_θ, determine the internal moments M caused only by the real loads.
- Since m or m_θ was assumed to act in the conventional positive direction, *it is important that positive* M acts in this *same* direction. This is necessary since positive or negative internal work depends upon the directional sense of load (defined by $\pm m$ or $\pm m_\theta$) and displacement (defined by $\pm M\,dx/EI$).

Virtual-Work Equation

- Apply the equation of virtual work to determine the desired displacement Δ or rotation θ. It is important to retain the algebraic sign of each integral calculated within its specified region.
- If the algebraic sum of all the integrals for the entire beam or frame is positive, Δ or θ is in the same direction as the virtual unit load or unit couple moment, respectively. If a negative value results, the direction of Δ or θ is opposite to that of the unit load or unit couple moment.

Example 10

- Determine the displacement Δ of point B of the steel beam shown in the figure below. Take $E = 200\ GPa, I = 500(10^6)mm^4$.



Solution 10

- Place a virtual unit load at B.
- No need to determine the reactions because x is chosen from B.
- Find the real moments at the same coordinate x.
- Virtual work equation for B is

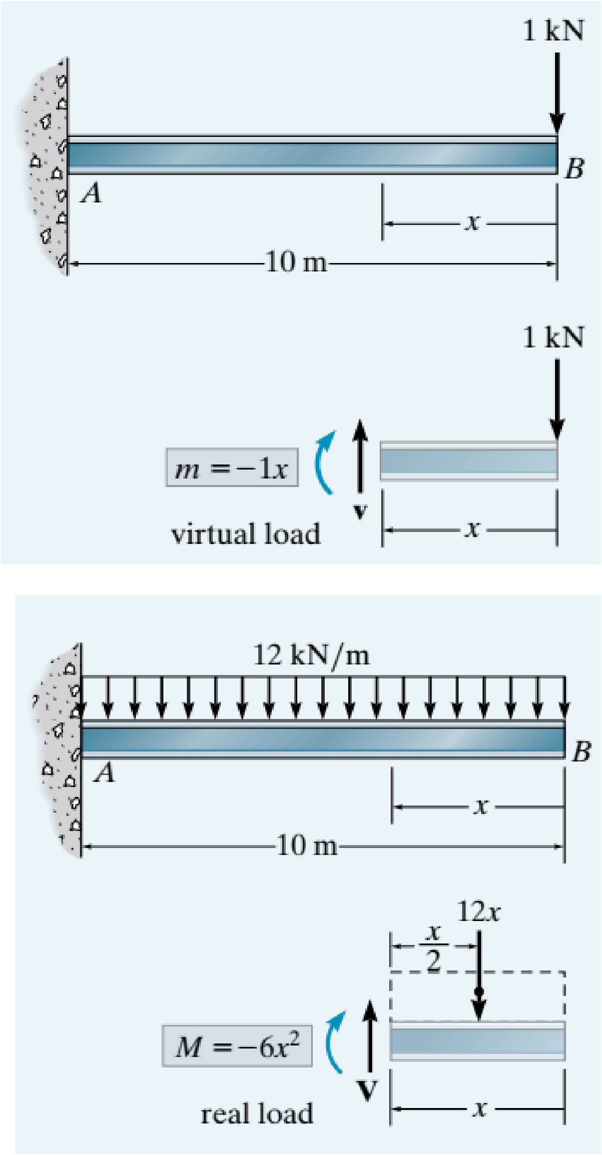
$$1\ \text{kN} \cdot \Delta_B = \int_0^L \frac{mM}{EI}dx = \int_0^{10} \frac{(-1x)(-6x^2)\,dx}{EI}$$

$$1\ \text{kN} \cdot \Delta_B = \frac{15(10^3)\ \text{kN}^2 \cdot \text{m}^3}{EI}$$

or

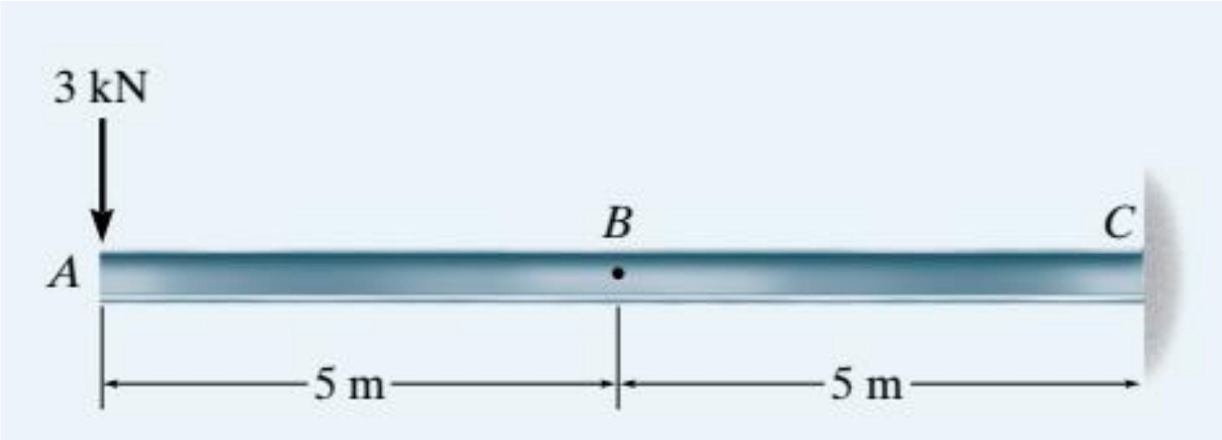
$$\Delta_B = \frac{15(10^3)\ \text{kN} \cdot \text{m}^3}{200(10^6)\ \text{kN/m}^2(500(10^6)\ \text{mm}^4)(10^{-12}\ \text{m}^4/\text{mm}^4)}$$

$$= 0.150\ \text{m} = 150\ \text{mm} \qquad \textit{Ans.}$$



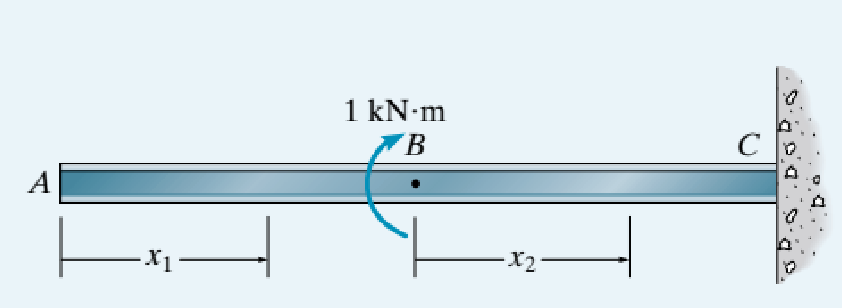
Example 11

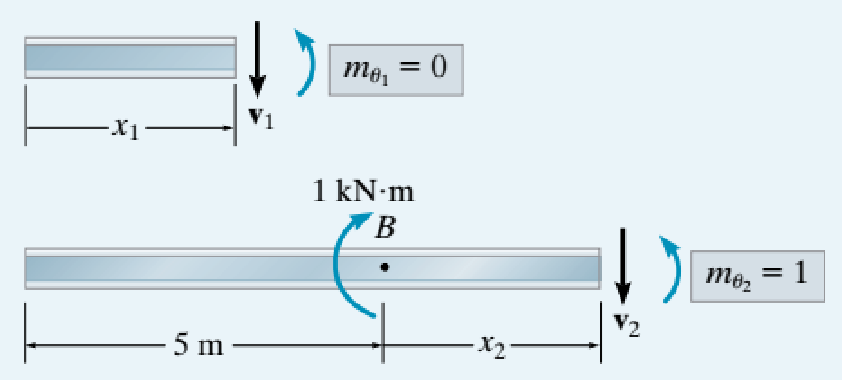
- Determine the slope θ of point B of the steel beam shown in the figure below. Take $E = 200\ GPa, I = 60(10^6)mm^4$.



Solution 11

- Place a virtual unit couple moment at B.
- 2 x coordinates are needed as shown. No need to determine the reactions because x is chosen from A.
- Find the real moments at the same coordinates.
- Virtual work equation for slope at B is





$$1 \cdot \theta_B = \int_0^L \frac{m_\theta M}{EI} dx$$

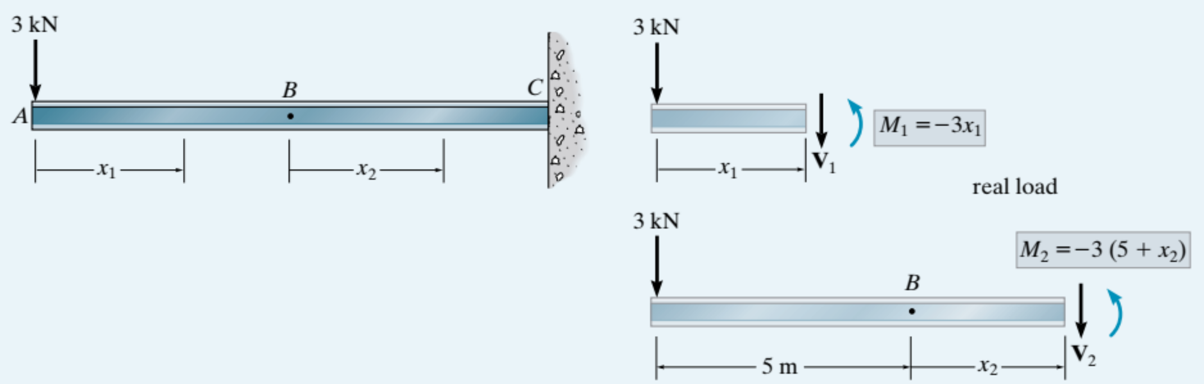
$$= \int_0^5 \frac{(0)(-3x_1)\, dx_1}{EI} + \int_0^5 \frac{(1)[-3(5 + x_2)]\, dx_2}{EI}$$

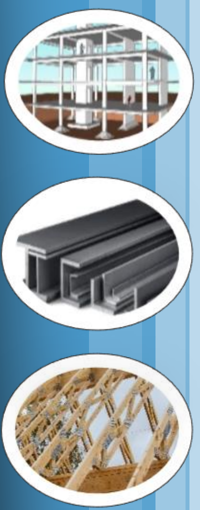
$$\theta_B = \frac{-112.5 \text{ kN} \cdot \text{m}^2}{EI}$$

$$(1 \text{ kN} \cdot \text{m}) \cdot \theta_B = \frac{-112.5 \text{ kN}^2 \cdot \text{m}^3}{200(10^6) \text{ kN/m}^2 \left[60(10^6) \text{ mm}^4\right](10^{-12} \text{ m}^4/\text{mm}^4)}$$

$$\theta_B = -0.00938 \text{ rad} \qquad \textit{Ans.}$$

The negative sign indicates θ_B is opposite to the direction of the virtual couple moment shown above

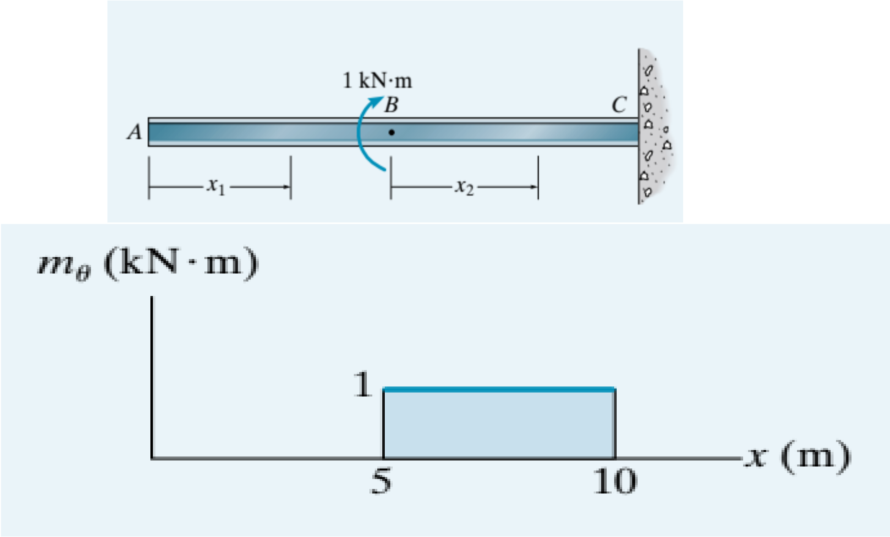




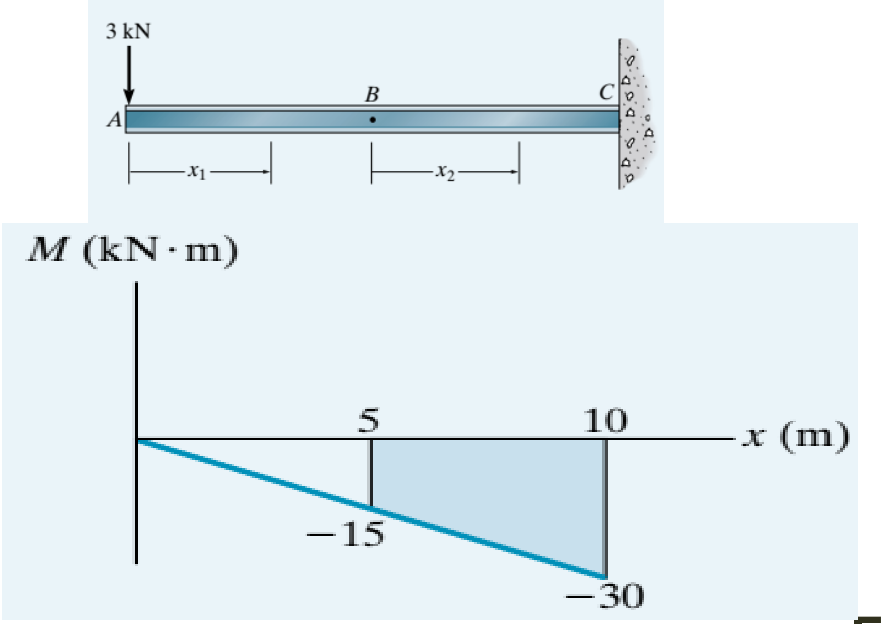
Solution 11 B

We can also evaluate the integrals $\int m_\theta M \, dx$ graphically

- Since there is no moment m for $0 \leq x \leq 5\ m$, we use only the shaded rectangular and trapezoidal areas to evaluate the integral.



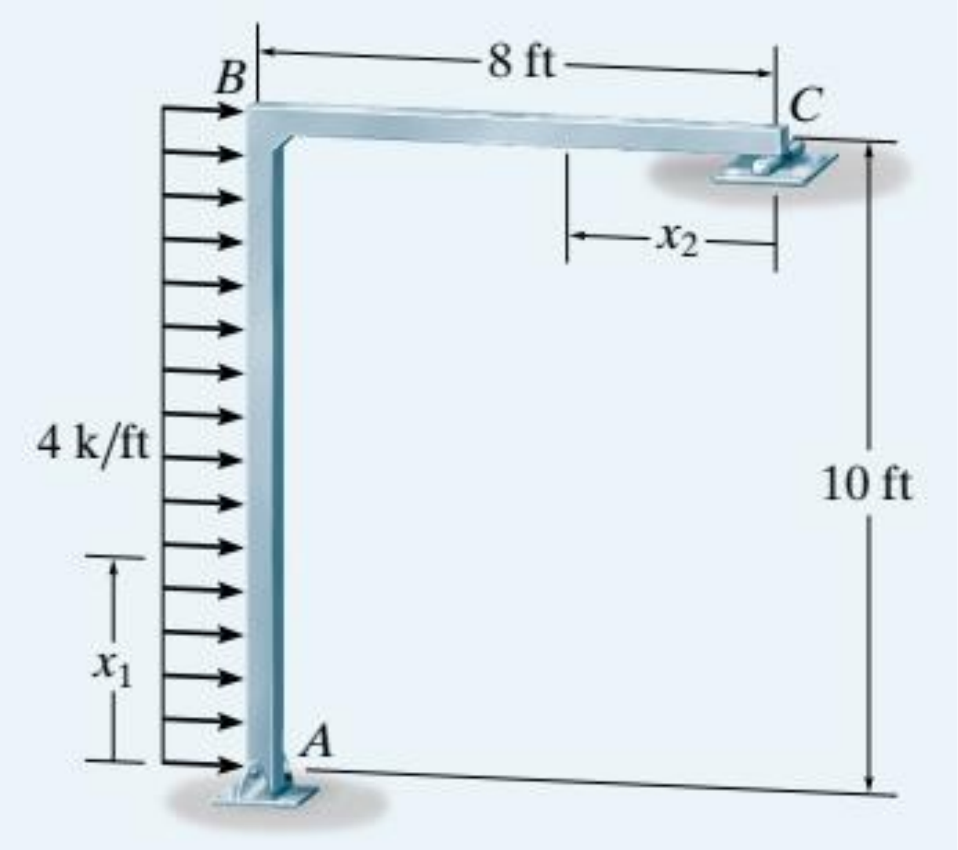
$$\int_5^{10} m_\theta M \, dx = \tfrac{1}{2} m_\theta (M_1 + M_2) L = \tfrac{1}{2}(1)(-15 - 30)5$$
$$= -112.5 \text{ kN}^2 \cdot \text{m}^3$$



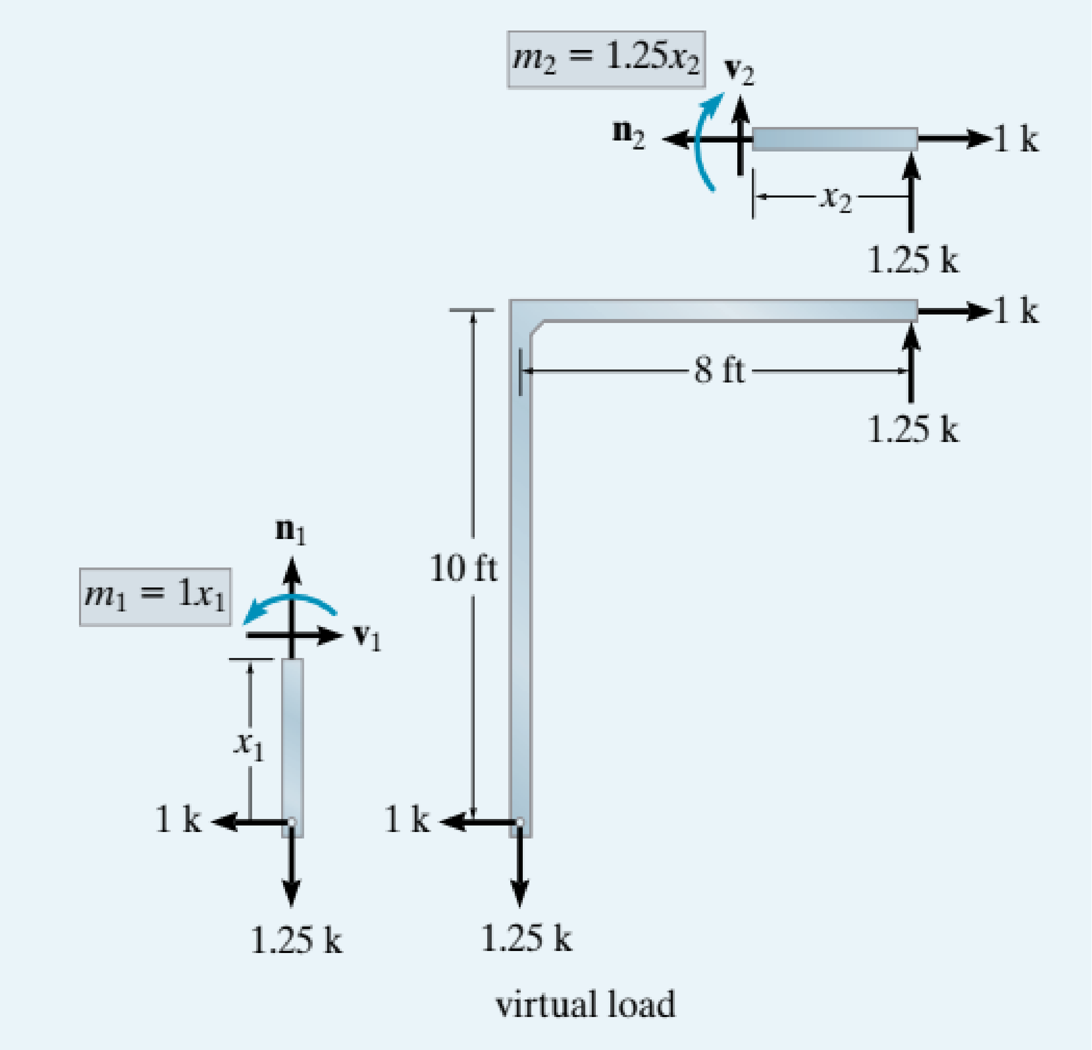
$$(1 \text{ kN} \cdot \text{m}) \cdot \theta_B = \frac{-112.5 \text{ kN}^2 \cdot \text{m}^3}{200(10^6) \text{ kN/m}^2 \left[60(10^6) \text{ mm}^4\right](10^{-12} \text{ m}^4/\text{mm}^4)}$$
$$\theta_B = -0.00938 \text{ rad} \qquad \textit{Ans.}$$

Example 12

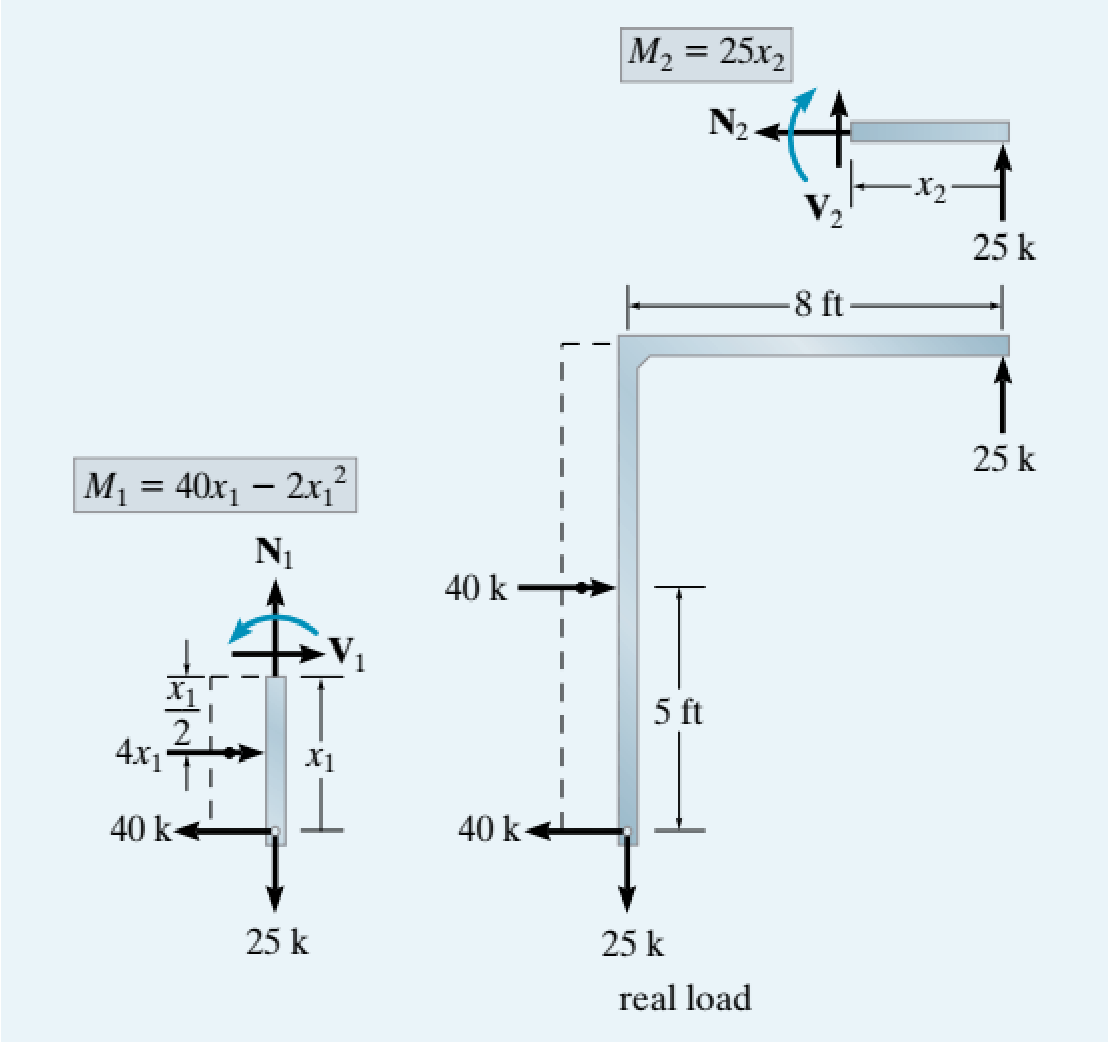
- Determine the horizontal displacement of point C on the frame figure below. Take $E = 29000\ ksi, I = 600\ in^4$.



Solution 12



$m_2 = 1.25x_2$
v_2
n_2
1 k
x_2
1.25 k
1 k
8 ft
1.25 k
10 ft
n_1
$m_1 = 1x_1$
v_1
x_1
1 k
1 k
1.25 k
1.25 k
virtual load



$M_2 = 25x_2$
N_2
V_2
x_2
25 k
8 ft
25 k
$M_1 = 40x_1 - 2x_1^2$
N_1
V_1
$\frac{x_1}{2}$
$4x_1$
x_1
40 k
5 ft
40 k
40 k
25 k
25 k
real load

Solution 12

$$1 \cdot \Delta_{C_h} = \int_0^L \frac{mM}{EI} dx = \int_0^{10} \frac{(1x_1)(40x_1 - 2x_1^2)dx_1}{EI} + \int_0^8 \frac{(1.25x_2)(25x_2)dx_2}{EI}$$

$$\Delta_{C_h} = \frac{8333.3}{EI} + \frac{5333.3}{EI} = \frac{13\,666.7 \text{ k} \cdot \text{ft}^3}{EI} \qquad (1)$$

$$\Delta_{C_h} = \frac{13\,666.7 \text{ k} \cdot \text{ft}^2}{\left[29(10^3) \text{ k/in}^2((12)^2 \text{ in}^2/\text{ft}^2)\right]\left[600 \text{ in } (\text{ft}^4/(12)^4 \text{ in}^4)\right]}$$

$$= 0.113 \text{ ft} = 1.36 \text{ in.} \qquad \textit{Ans.}$$

Method of Virtual Work: Beams and Frames

- **Axial Load.** Frame members can be subjected to axial loads, and the virtual strain energy caused by these loadings has been established. For members having a constant cross-sectional area, we have

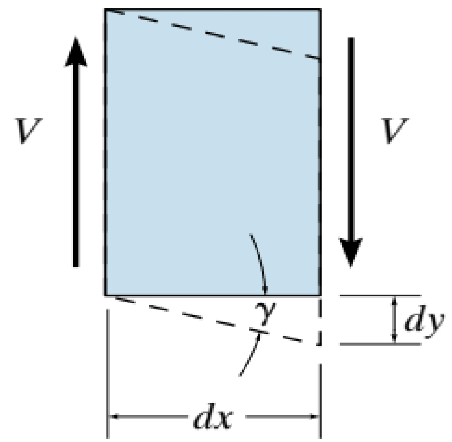
$$U_n = \frac{nNL}{AE}$$

where

n = internal virtual axial load caused by the external virtual unit load.
N = internal axial force in the member caused by the real loads.
E = modulus of elasticity for the material.
A = cross-sectional area of the member.
L = member's length.

Method of Virtual Work: Beams and Frames

- **Shear.** In order to determine the virtual strain energy in a beam due to shear, we will consider the beam element dx shown in the figure. The shearing distortion dy of the element as caused by the real loads is $dy = \gamma\, dx$ where$\gamma = \tau/G$ and $\tau = K(V/A)$. $dU_s = v dy = v(KV/GA)\, dx$.



$$U_s = \int_0^L K\left(\frac{vV}{GA}\right) dx$$

where

v = internal virtual shear in the member, expressed as a function of x and caused by the external virtual unit load.

V = internal shear in the member, expressed as a function of x and caused by the real loads.

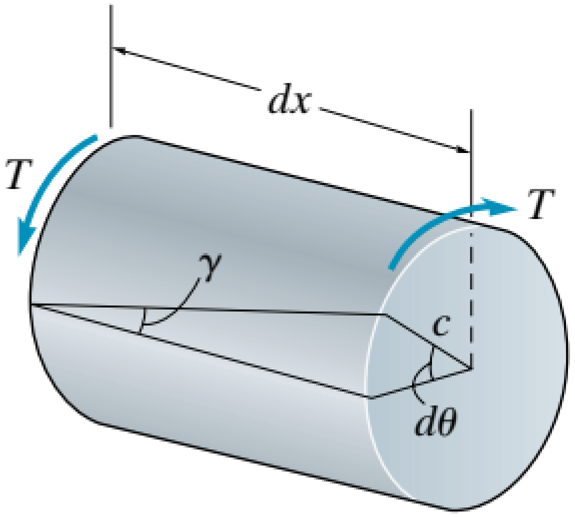
A = cross-sectional area of the member.

K = form factor for the cross-sectional area:

- $K = 1.2$ for rectangular cross sections.
- $K = 10/9$ for circular cross sections.
- $K \approx 1$ for wide-flange and I-beams, where A is the area of the web.

G = shear modulus of elasticity for the material.

Method of Virtual Work: Beams and Frames



- **Torsion.** If the member has a circular cross-sectional area, no warping of its cross section will occur when it is loaded. The applied torque **T** to an element dx will cause shear strain of $\gamma = (cd\theta)/dx$. Provided linear elastic, $\gamma = \tau/G$, $\tau = Tc/J$, after manipulations, $d\theta = (T/GJ)dx$. Thus, $U_t = td\theta = t(T/GJ)dx$.

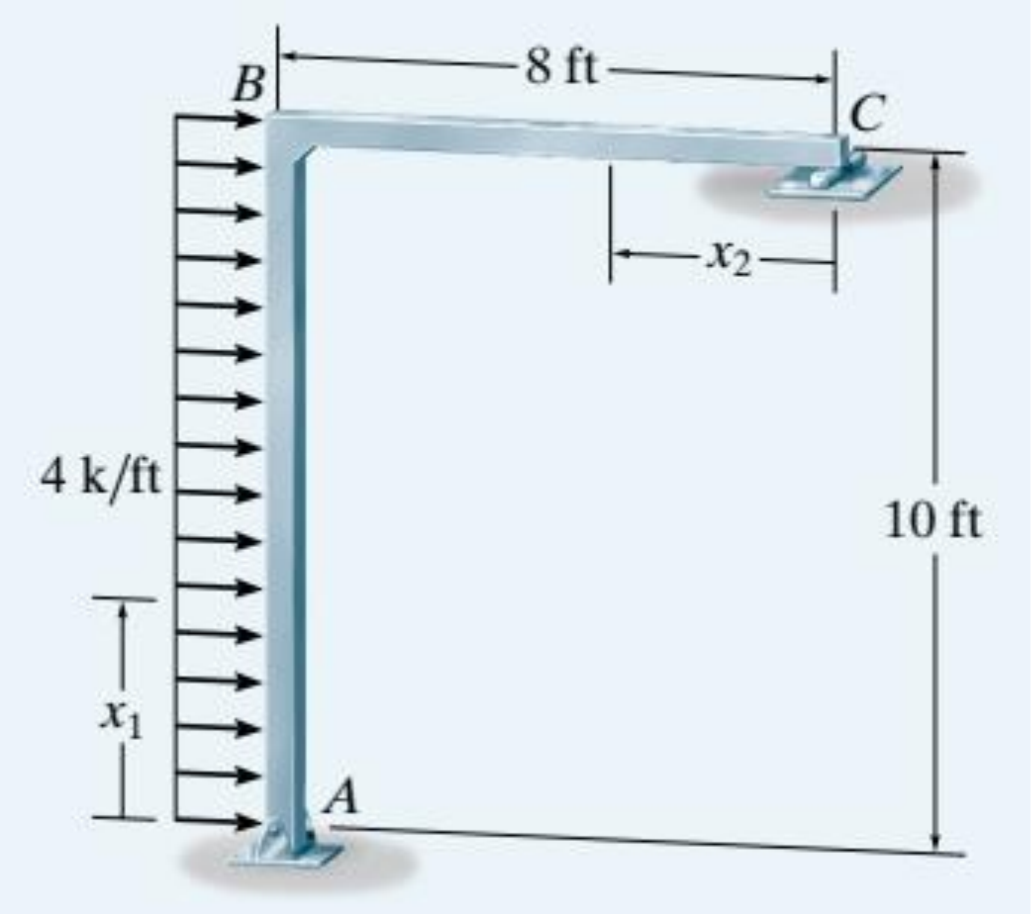
$$U_t = \frac{tTL}{GJ}$$

where

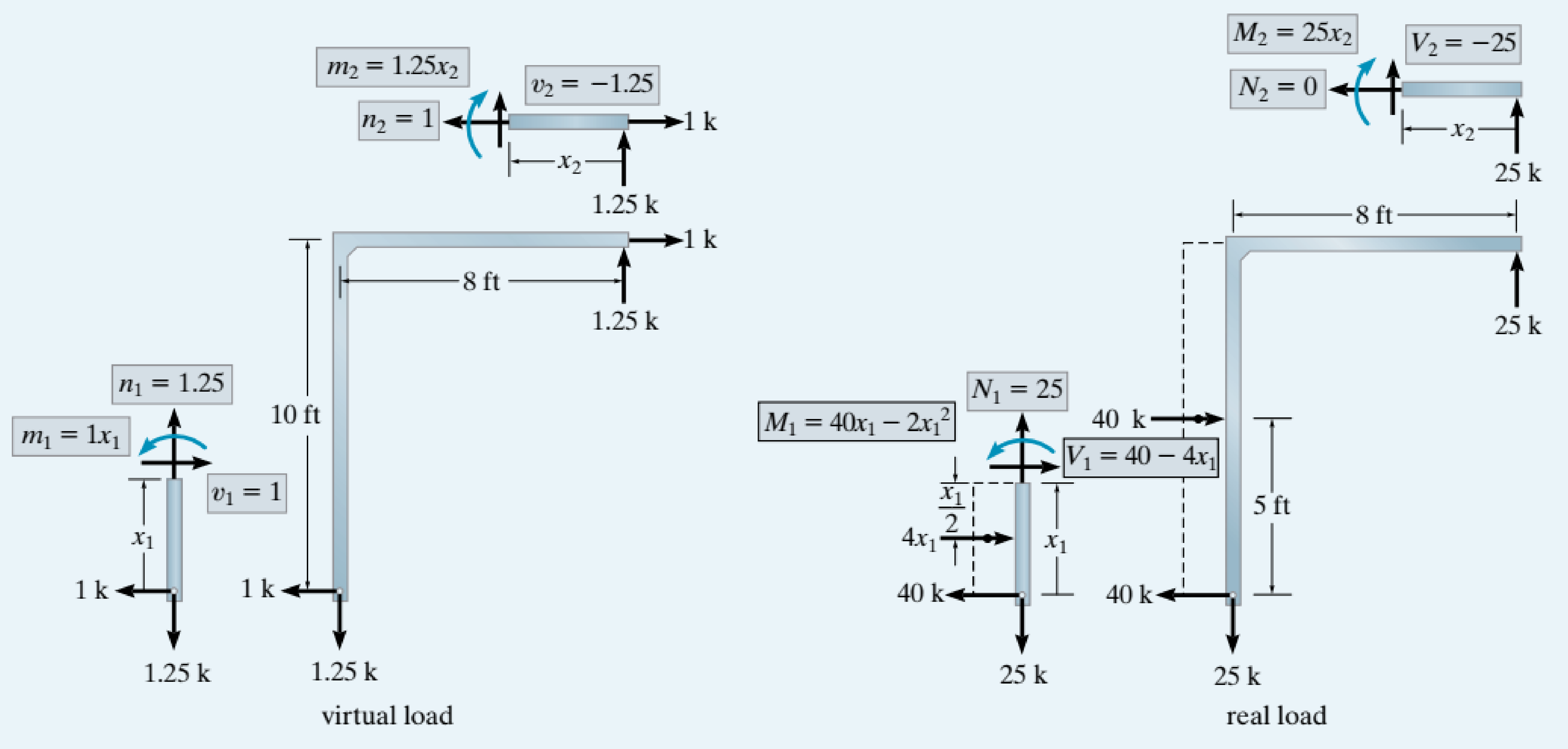
t = internal virtual torque caused by the external virtual unit load.
T = internal torque in the member caused by the real loads.
G = shear modulus of elasticity for the material.
J = polar moment of inertia for the cross section, $J = \pi c^4/2$, where c is the radius of the cross-sectional area.
L = member's length.

Example 13

- Determine the horizontal displacement of point C on the frame shown in the figure. Take $E = 29(10^3)ksi, G = 12(10^3)ksi,\ \ I = 600\ in^4$, and A = 80 in^2 for both members. The cross-sectional area is rectangular. Include the internal strain energy due to axial load and shear.



Solution 13



$m_2 = 1.25x_2$
$v_2 = -1.25$
$n_2 = 1$
1 k
x_2
1.25 k
1 k
8 ft
1.25 k
10 ft
$n_1 = 1.25$
$m_1 = 1x_1$
$v_1 = 1$
x_1
1 k
1 k
1.25 k
1.25 k
virtual load
$M_2 = 25x_2$
$V_2 = -25$
$N_2 = 0$
x_2
25 k
8 ft
25 k
$N_1 = 25$
$M_1 = 40x_1 - 2x_1^2$
40 k
$V_1 = 40 - 4x_1$
$\frac{x_1}{2}$
$4x_1$
x_1
5 ft
40 k
40 k
25 k
25 k
real load

Solution 13

Bending. The virtual strain energy due to bending has been determined in Example 9.10. There it was shown that

$$U_b = \int_0^L \frac{mM\,dx}{EI} = \frac{13\,666.7\text{ k}^2\cdot\text{ft}^3}{EI} = \frac{13\,666.7\text{ k}^2\cdot\text{ft}^3\,(12^3\text{ in}^3/1\text{ ft}^3)}{[29(10^3)\text{ k/in}^2](600\text{ in}^4)} = 1.357\text{ in.}\cdot\text{k}$$

Axial load. From the data in Fig. 9–25*b* and 9–25*c*, we have

$$U_a = \sum \frac{nNL}{AE}$$

$$= \frac{1.25\text{ k}(25\text{ k})(120\text{ in.})}{80\text{ in}^2[29(10^3)\text{ k/in}^2]} + \frac{1\text{ k}(0)(96\text{ in.})}{80\text{ in}^2[29(10^3)\text{ k/in}^2]}$$

$$= 0.001616\text{ in.}\cdot\text{k}$$

Shear. Applying Eq. 9–25 with $K = 1.2$ for rectangular cross sections, and using the shear functions shown in Fig. 9–25*b* and 9–25*c*, we have

$$U_s = \int_0^L K\left(\frac{vV}{GA}\right)dx$$

$$= \int_0^{10} \frac{1.2(1)(40 - 4x_1)\,dx_1}{GA} + \int_0^{8} \frac{1.2(-1.25)(-25)\,dx_2}{GA}$$

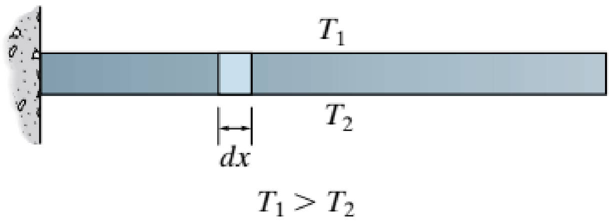
$$= \frac{540\text{ k}^2\cdot\text{ft}(12\text{ in./ft})}{[12(10^3)\text{ k/in}^2](80\text{ in}^2)} = 0.00675\text{ in.}\cdot\text{k}$$

Applying the equation of virtual work, we have

$$1\text{ k}\cdot\Delta_{C_h} = 1.357\text{ in.}\cdot\text{k} + 0.001616\text{ in.}\cdot\text{k} + 0.00675\text{ in.}\cdot\text{k}$$

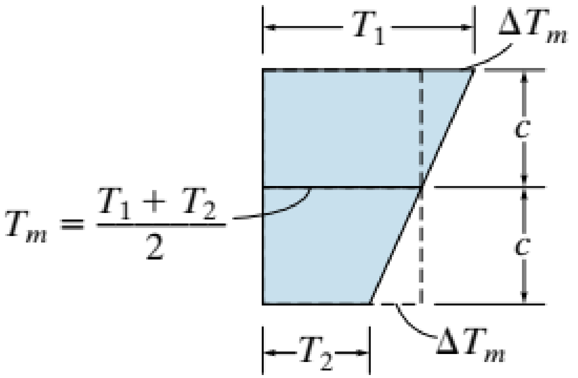
$$\Delta_{C_h} = 1.37\text{ in.} \qquad \textit{Ans.}$$

Including the effects of shear and axial load contributed only a 0.6% increase in the answer to that determined only from bending.



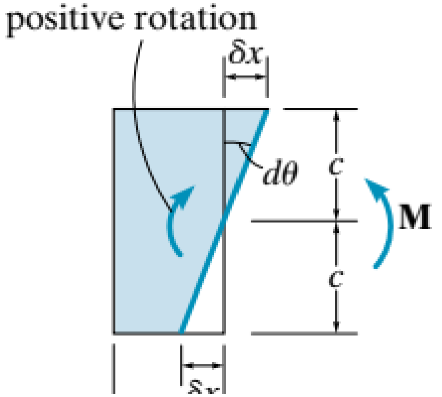
$$T_m = (T_1 + T_2)/2$$

$$\Delta T_m = T_1 - T_m = T_m - T_2$$



- **Temperature.** A structural member can be subjected to a temperature difference across its depth, as in the case of the beam shown in figure.

$$d\theta = \frac{\alpha\, \Delta T_m\, dx}{c} \qquad U_{temp} = \int_0^L \frac{m\alpha\, \Delta T_m\, dx}{c}$$



where

m = internal virtual moment in the beam expressed as a function of x and caused by the external virtual unit load or unit couple moment.

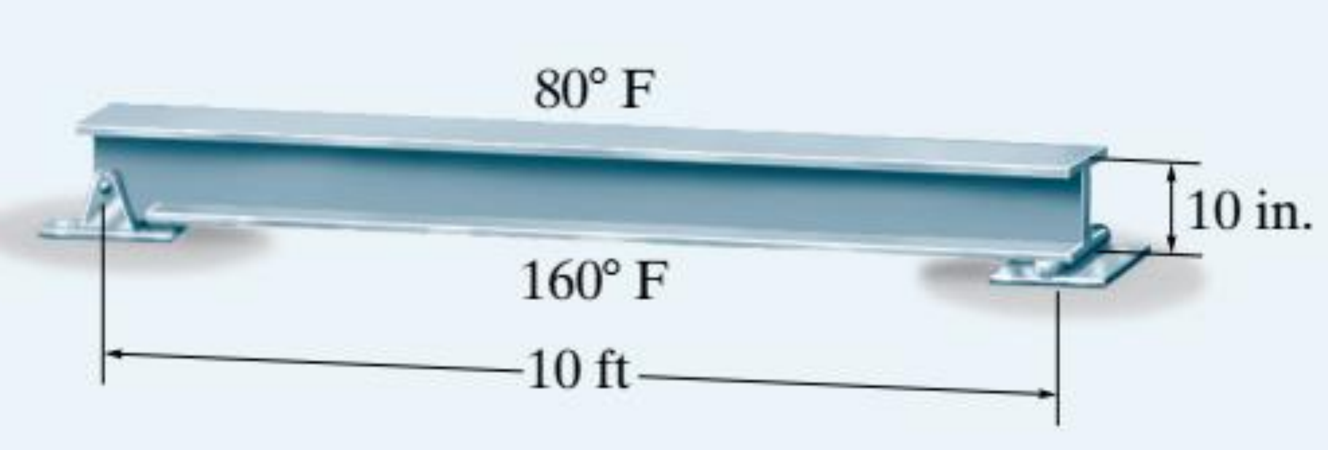
α = coefficient of thermal expansion.

ΔT_m = temperature difference between the mean temperature and the temperature at the top or bottom of the beam.

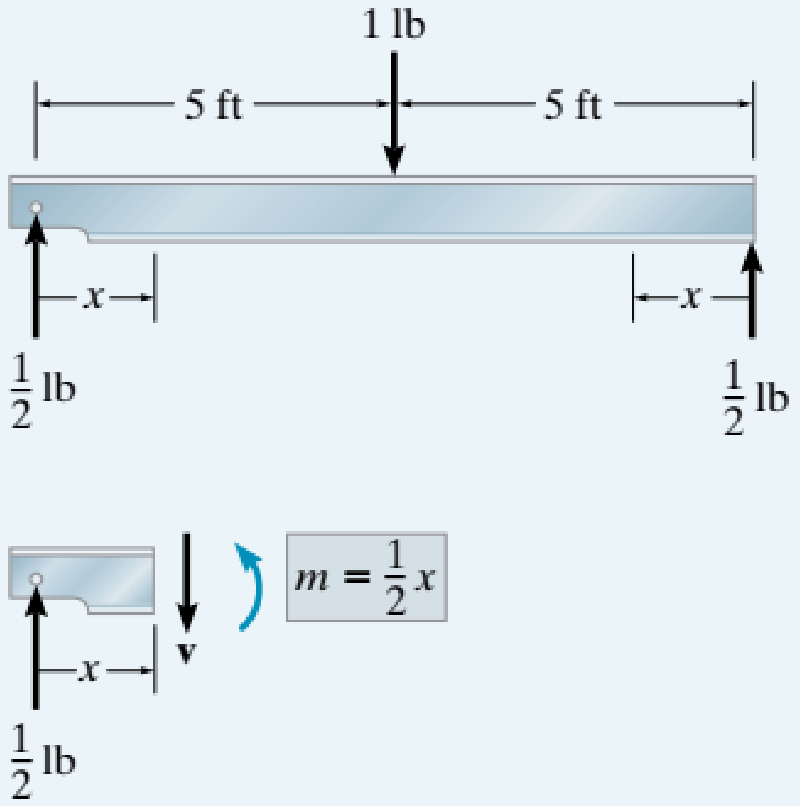
c = mid-depth of the beam.

Example 14

- The beam shown in the figure is used in a building subjected to two different thermal environments. If the temperature at the top surface of the beam is 80°F and at the bottom surface is 160°F, determine the vertical deflection of the beam at its midpoint due to the temperature gradient. Take $\alpha = 6.5(10^{-6})/{}^{0}F$.



Solution 13



virtual load

Since the deflection at the center of the beam is to be determined, a virtual unit load is placed there and the internal virtual moment in the beam is calculated, Fig. 9–26*b*.

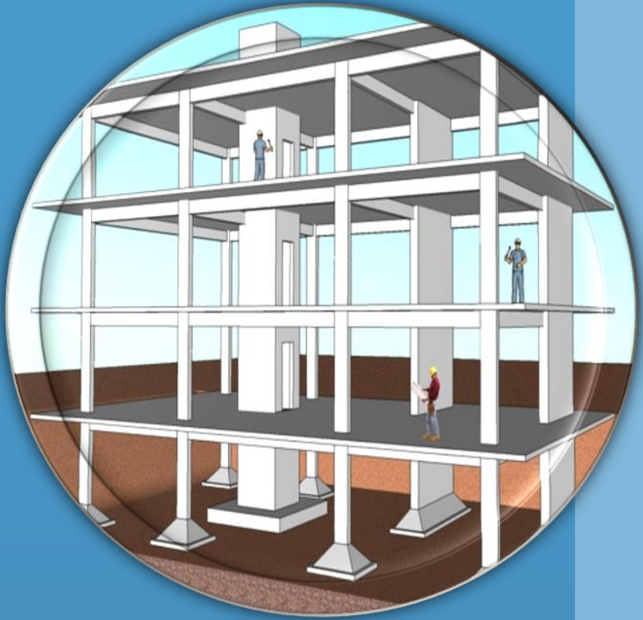
The mean temperature at the center of the beam is $(160° + 80°)/2 = 120°F$, so that for application of Eq. 9–27, $\Delta T_m = 120°F - 80°F = 40°F$. Also, $c = 10 \text{ in.}/2 = 5 \text{ in.}$ Applying the principle of virtual work, we have

$$1 \text{ lb} \cdot \Delta_{C_v} = \int_0^L \frac{m\alpha \, \Delta T_m \, dx}{c}$$

$$= 2\int_0^{60 \text{ in.}} \frac{\left(\frac{1}{2}x\right)6.5(10^{-6})/°F(40°F)}{5 \text{ in.}} dx$$

$$\Delta_{C_v} = 0.0936 \text{ in.} \qquad \textit{Ans.}$$

The result indicates a very negligible deflection.



Thank You!