UNIVERSITY OF ZAMBIA
School of Engineering
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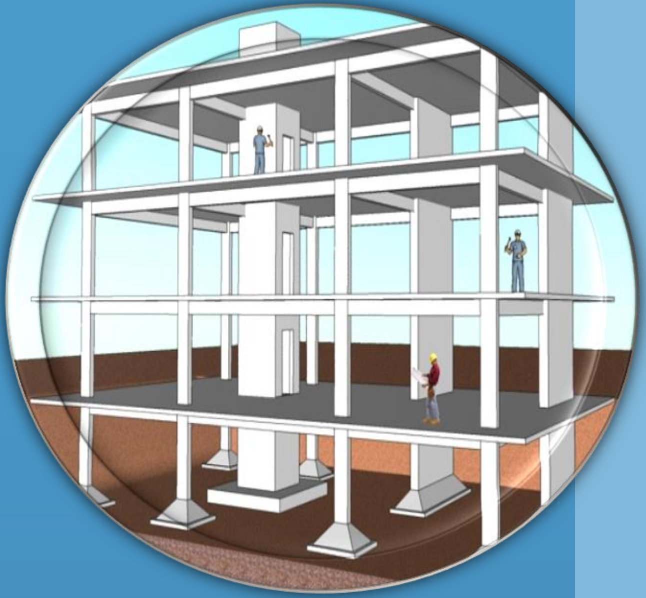
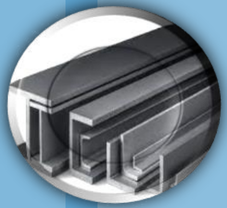


CEE 3222: THEORY OF STRUCTURES

Lecture 3.3

ENERGY METHODS - CASTIGLIANO's THEOREMS



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Contents

Introduction

- All energy methods are based on the conservation of energy principle, which states that the work done by all external forces acting on the structure, U_e, is transformed into internal work or strain energy. U_i, developed in the members when the structure deforms.

$$U_e = U_i$$

Castigliano's Theorem

- This method is also called ***Castigliano's second theorem***, or the **method of least work.**
- *It states that the displacement (slope) of a point is equal to the first partial derivative of the strain energy in the structure with respect to a force (couple moment) acting at the point and in the direction of displacement*

$$\Delta_i = \frac{\partial U_i}{\partial P_i}$$

Assumptions on structures.

1. **constant temperature**
2. **unyielding supports**
3. **linear elastic material response.**

- The equation is a statement regarding the structure's compatibility.
- The above derivation requires that only conservative forces be considered for the analysis.
- These forces do work that is independent of the path and therefore create no energy loss.

Castigliano's Theorem for Trusses

- Recall that the strain energy for an axial force is given by

$$U_i = \frac{N^2 L}{2AE}$$

- Substituting this in $\Delta_i = \frac{\partial U_i}{\partial P_i}$

- We get formula using the theorem

$$\Delta = \sum N \left(\frac{\partial N}{\partial P}\right) \frac{L}{AE}$$

where

Δ = external joint displacement of the truss.

P = external force applied to the truss joint in the direction of Δ.

N = internal force in a member caused by *both* the force P and the loads on the truss.

L = length of a member.

A = cross-sectional area of a member.

E = modulus of elasticity of a member.

- Each member force N must be expressed as a function of P.

Castigliano's Theorem for Trusses

Procedure for Analysis

The following procedure provides a method that may be used to determine the displacement of any joint of a truss using Castigliano's theorem.

External Force P

- Place a force P on the truss at the joint where the desired displacement is to be determined. This force is assumed to have a *variable magnitude* in order to obtain the change $\partial N/\partial P$. Be sure **P** is directed along the line of action of the displacement.

Internal Forces N

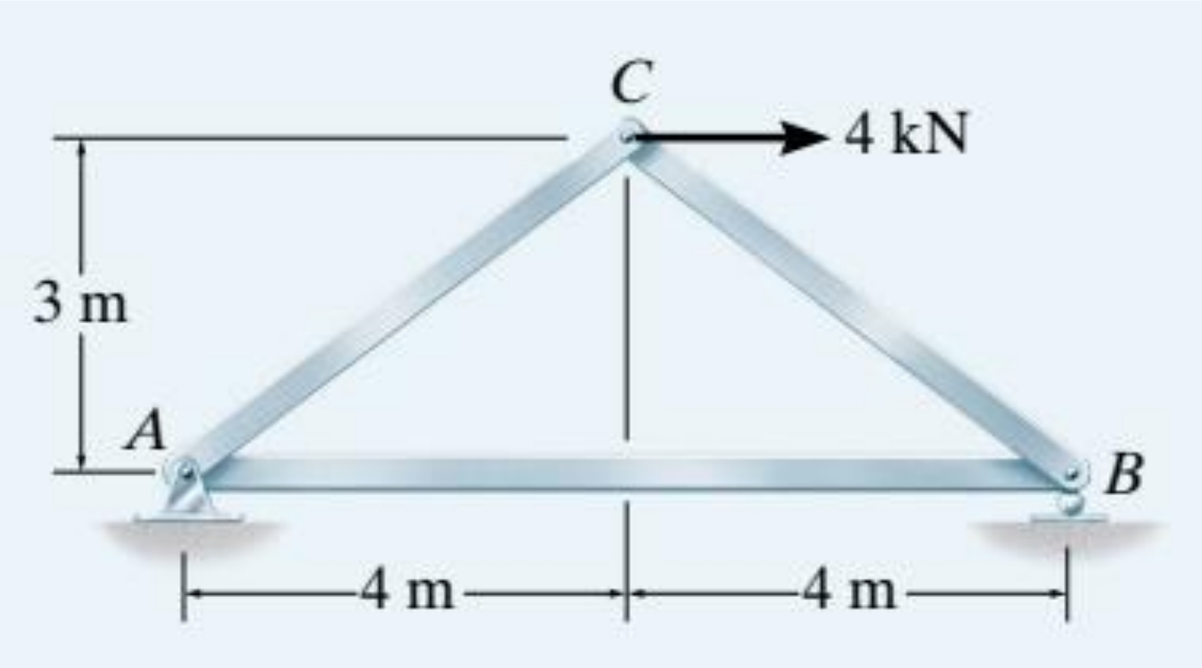
- Determine the force N in each member caused by both the real (numerical) loads and the variable force P. Assume tensile forces are positive and compressive forces are negative.
- Compute the respective partial derivative $\partial N/\partial P$ for each member.
- After N and $\partial N/\partial P$ have been determined, assign P its numerical value if it has replaced a real force on the truss. Otherwise, set P equal to zero.

Castigliano's Theorem

- Apply Castigliano's theorem to determine the desired displacement Δ. It is important to retain the algebraic signs for corresponding values of N and $\partial N/\partial P$ when substituting these terms into the equation.
- If the resultant sum $\Sigma N(\partial N/\partial P)L/AE$ is positive, Δ is in the same direction as P. If a negative value results, Δ is opposite to P.

Example 1

- Determine the vertical displacement of joint C of the truss shown in the figure. The cross-sectional area of each member is $A = 400mm^2$ and $E = 200\ GPa$



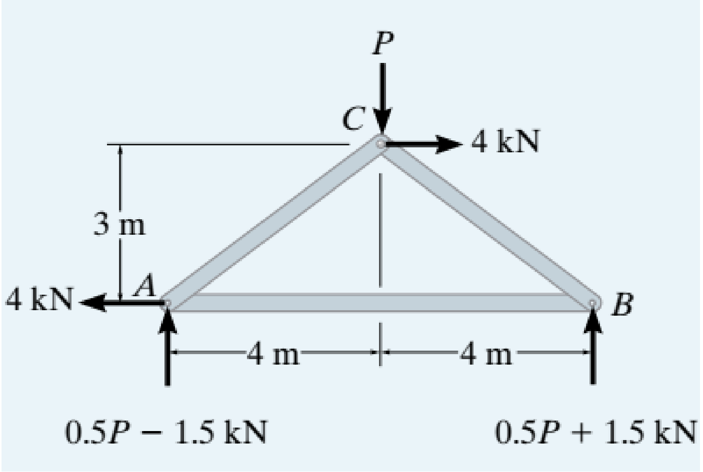
Solution 1

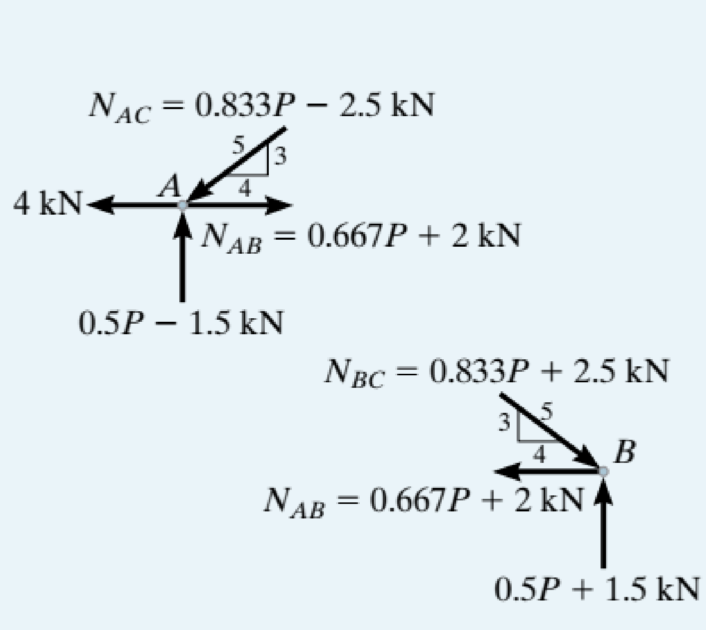
- A vertical force P is applied to the truss at joint C, since this is where the vertical displacement is to be determined
- Since P does not actually exist as a real load on the truss, we require P = 0 in the table below

Member	N	$\frac{\partial N}{\partial P}$	$N(P=0)$	L	$N\left(\frac{\partial N}{\partial P}\right)L$
AB	$0.667P + 2$	0.667	2	8	10.67
AC	$-(0.833P - 2.5)$	-0.833	2.5	5	-10.42
BC	$-(0.833P + 2.5)$	-0.833	-2.5	5	10.42
					$\Sigma = 10.67\ \text{kN}\cdot\text{m}$

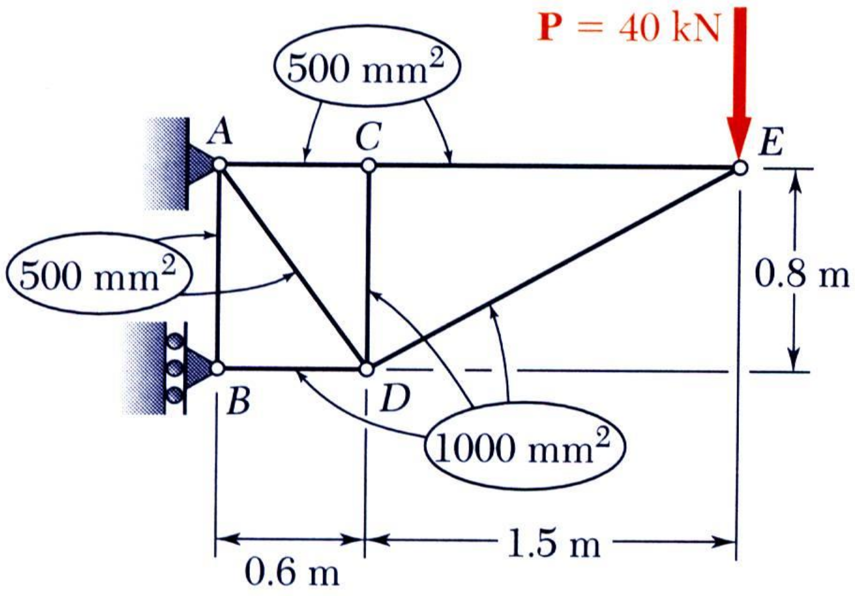
$$\Delta_{C_v} = \sum N\left(\frac{\partial N}{\partial P}\right)\frac{L}{AE} = \frac{10.67\ \text{kN}\cdot\text{m}}{AE}$$

$$\Delta_{C_v} = \frac{10.67(10^3)\ \text{N}\cdot\text{m}}{400(10^{-6})\ \text{m}^2(200(10^9)\ \text{N/m}^2)} = 0.000133\ \text{m} = 0.133\ \text{mm} \quad \textit{Ans.}$$





Example 2 + Solution



Members of the truss shown consist of sections of aluminum pipe with the cross-sectional areas indicated. Using *E* = 73 GPa, determine the vertical deflection of the joint *C* caused by the load *P*.

SOLUTION:

- **For application of Castigliano's theorem, introduce a dummy vertical load *Q* at *C*. Find the reactions at *A* and *B* due to the dummy load from a free-body diagram of the entire truss.**

- **Apply the method of joints to determine the axial force in each member due to *Q*.**

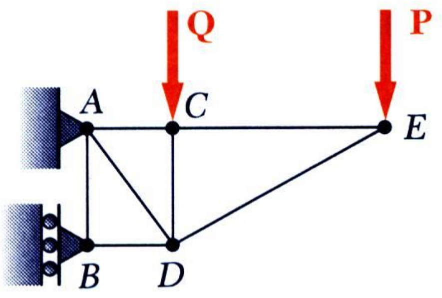
- **evaluate the derivative with respect to *Q* of the strain energy of the truss due to the loads *P* and *Q*.**

- **Setting *Q* = 0, evaluate the derivative which is equivalent to the desired displacement at *C*.**

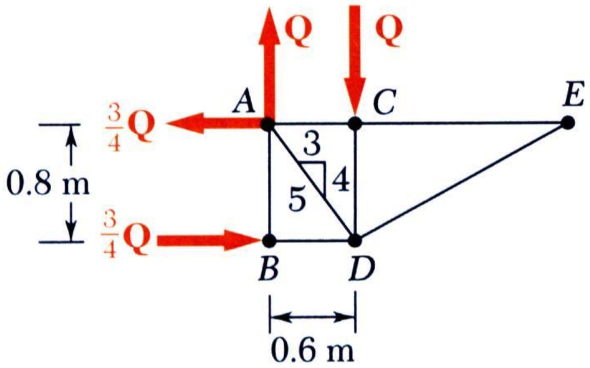
Solution 2

SOLUTION:

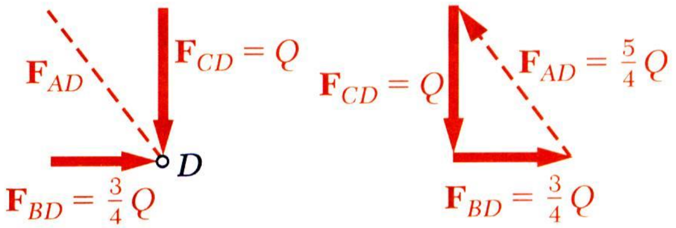
- **Find the reactions at *A* and *B* due to a dummy load *Q* at *C* from a free-body diagram of the entire truss.**



$$A_x = -\tfrac{3}{4}Q \qquad A_y = Q \qquad B = \tfrac{3}{4}Q$$



- **Apply the method of joints to determine the axial force in each member due to *Q*.**

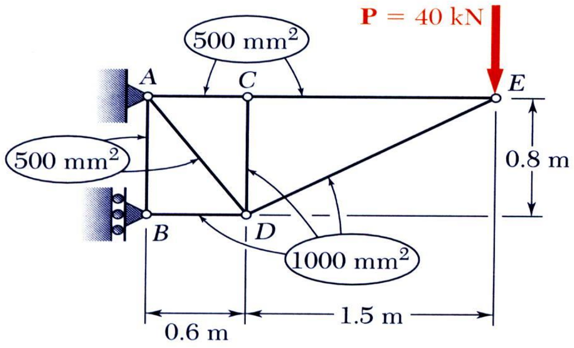


$$F_{CE} = F_{DE} = 0$$

$$F_{AC} = 0;\ F_{CD} = -Q$$

$$F_{AB} = 0;\ F_{BD} = -\frac{3}{4}Q$$

Solution 2



Member	F_i	$\partial F_i/\partial Q$	L_i, m	A_i, m²	$\left(\frac{F_iL_i}{A_i}\right)\frac{\partial F_i}{\partial Q}$
AB	0	0	0.8	500×10^{-6}	0
AC	$+15P/8$	0	0.6	500×10^{-6}	0
AD	$+5P/4 + 5Q/4$	$\frac{5}{4}$	1.0	500×10^{-6}	$+3125P + 3125Q$
BD	$-21P/8 - 3Q/4$	$-\frac{3}{4}$	0.6	1000×10^{-6}	$+1181P + 338Q$
CD	$-Q$	-1	0.8	1000×10^{-6}	$+ 800Q$
CE	$+15P/8$	0	1.5	500×10^{-6}	0
DE	$-17P/8$	0	1.7	1000×10^{-6}	0

- **Combine with the results of Sample Problem 11.4 to evaluate the derivative with respect to Q of the strain energy of the truss due to the loads P and Q.**

$$y_C = \sum \left(\frac{F_i L_i}{A_i E}\right)\frac{\partial F_i}{\partial Q} = \frac{1}{E}(4306P + 4263Q)$$

- **Setting $Q = 0$, evaluate the derivative which is equivalent to the desired displacement at C.**

$$y_C = \frac{4306\left(40 \times 10^3 N\right)}{73 \times 10^9 \text{ Pa}}$$

$$y_C = 2.36 \text{ mm} \downarrow$$

Castigliano's Theorem for Beams and Frames

- The internal bending strain energy for a beam or frame is given by $U_i = \int M^2 dx/2EI$. Substituting this equation into $\Delta_i = \partial U_i/\partial P_i$ and omitting the subscript i, we have

$$\Delta = \frac{\partial}{\partial P}\int_0^L \frac{M^2\,dx}{2EI}$$

$$\Delta = \int_0^L M\left(\frac{\partial M}{\partial P}\right)\frac{dx}{EI}$$

where

Δ = external displacement of the point caused by the real loads acting on the beam or frame.

P = external force applied to the beam or frame in the direction of Δ.

M = internal moment in the beam or frame, expressed as a function of x and caused by both the force P and the real loads on the beam.

E = modulus of elasticity of beam material.

I = moment of inertia of cross-sectional area computed about the neutral axis.

Castigliano's Theorem for Beams and Frames

- If the slope θ at a point is to be determined, we must find the partial derivative of the internal moment M with respect to an external couple moment M' acting at the point.

$$\theta = \int_0^L M\left(\frac{\partial M}{\partial M'}\right)\frac{dx}{EI}$$

- For shear and torsion, we have the following and their partial derivative.

$$U_s = K\int_0^L \frac{V^2\,dx}{2AG} \qquad \frac{\partial U_s}{\partial P} = \int_0^L \frac{V}{AG}\left(\frac{\partial V}{\partial P}\right)dx$$

$$U_t = \int_0^L \frac{T^2\,dx}{2JG} \qquad \frac{\partial U_t}{\partial P} = \int_0^L \frac{T}{JG}\left(\frac{\partial T}{\partial P}\right)dx$$

Castigliano's Theorem for Beams and Frames

Procedure for Analysis

The following procedure provides a method that may be used to determine the deflection and/or slope at a point in a beam or frame using Castigliano's theorem.

External Force P or Couple Moment M'

- Place a force **P** on the beam or frame at the point and in the direction of the desired displacement.
- If the slope is to be determined, place a couple moment $\mathbf{M}'$ at the point.
- It is assumed that both P and M' have a *variable magnitude* in order to obtain the changes $\partial M/\partial P$ or $\partial M/\partial M'$.

Internal Moments M

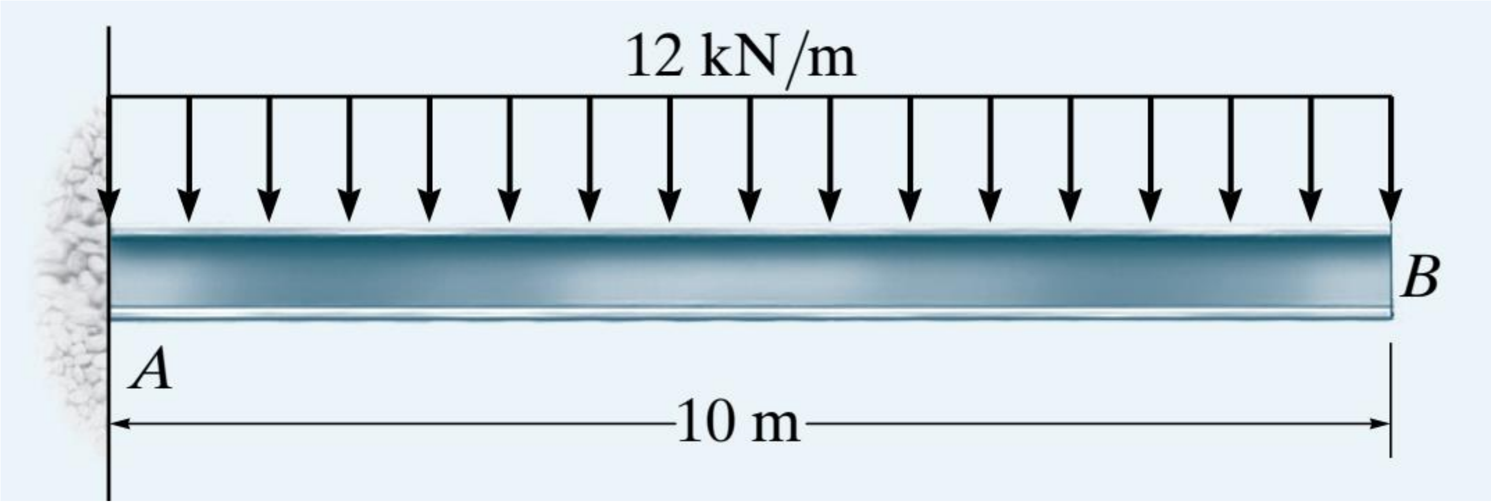
- Establish appropriate x coordinates that are valid within regions of the beam or frame where there is no discontinuity of force, distributed load, or couple moment.
- Calculate the internal moment M as a function of P or M' and each x coordinate. Also, compute the partial derivative $\partial M/\partial P$ or $\partial M/\partial M'$ for each coordinate x.
- After M and $\partial M/\partial P$ or $\partial M/\partial M'$ have been determined, assign P or M' its numerical value if it has replaced a real force or couple moment. Otherwise, set P or M' equal to zero.

Castigliano's Theorem

- Apply Eq. 9–28 or 9–29 to determine the desired displacement Δ or slope θ. It is important to retain the algebraic signs for corresponding values of M and $\partial M/\partial P$ or $\partial M/\partial M'$.
- If the resultant sum of all the definite integrals is positive, Δ or θ is in the same direction as **P** or $\mathbf{M}'$.

Example 3

- Determine the displacement of point B of the beam shown in the figure. Take $E = 200GPa$, $I = 500(10^6)mm^4$.



Solution 3

- A vertical force P is placed on the beam at B as shown in the figure.

Internal Moments *M*. A single x coordinate is needed for the solution, since there are no discontinuities of loading between A and B. Using the method of sections,

$$\circlearrowleft + \Sigma M = 0; \qquad -M - (12x)\left(\frac{x}{2}\right) - Px = 0$$

$$M = -6x^2 - Px \qquad \frac{\partial M}{\partial P} = -x$$

Setting $P = 0$, its actual value, yields

$$M = -6x^2 \qquad \frac{\partial M}{\partial P} = -x$$

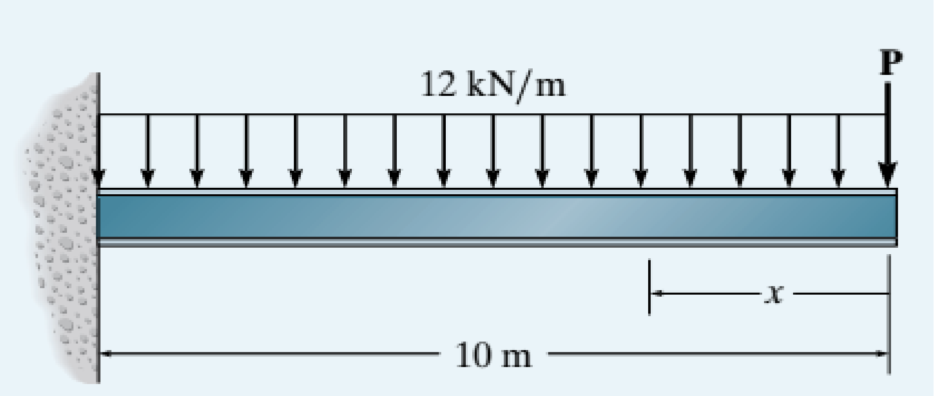
Castigliano's Theorem. Applying Eq. 9–28, we have

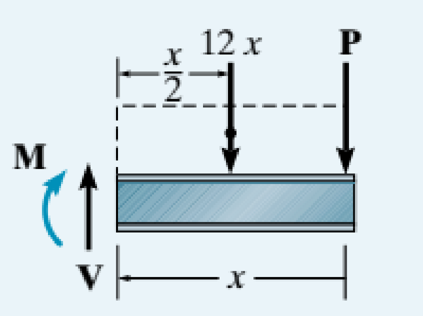
$$\Delta_B = \int_0^L M\left(\frac{\partial M}{\partial P}\right)\frac{dx}{EI} = \int_0^{10} \frac{(-6x^2)(-x)\,dx}{EI} = \frac{15(10^3)\text{ kN}\cdot\text{m}^3}{EI}$$

or

$$\Delta_B = \frac{15(10^3)\text{ kN}\cdot\text{m}^3}{200(10^6)\text{ kN/m}^2[500(10^6)\text{ mm}^4](10^{-12}\text{ m}^4/\text{mm}^4)}$$

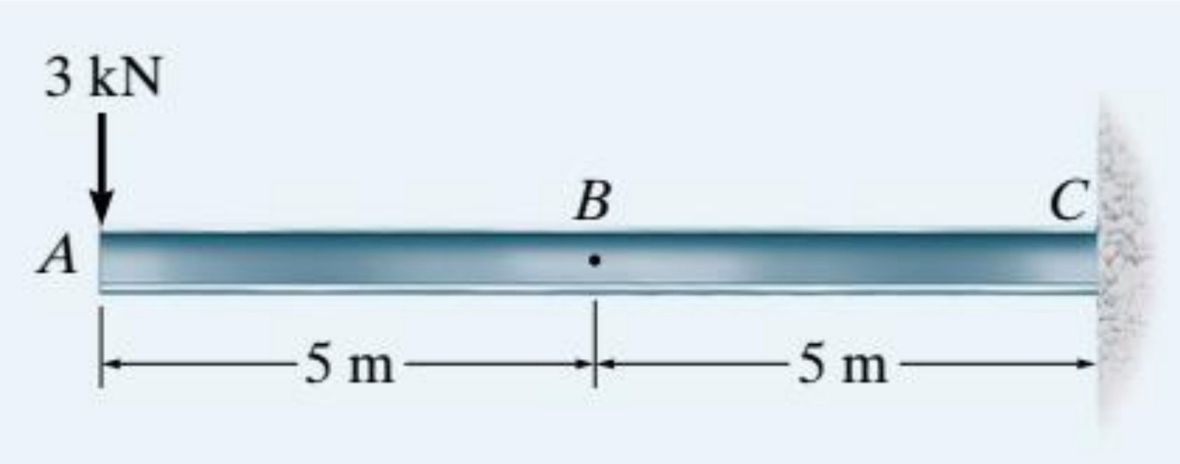
$$= 0.150\text{ m} = 150\text{ mm} \qquad \textit{Ans.}$$





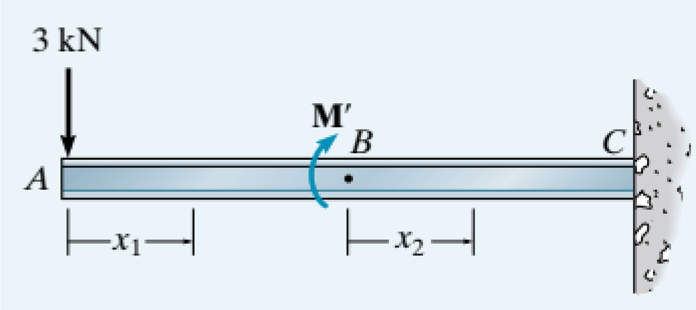
Example 4

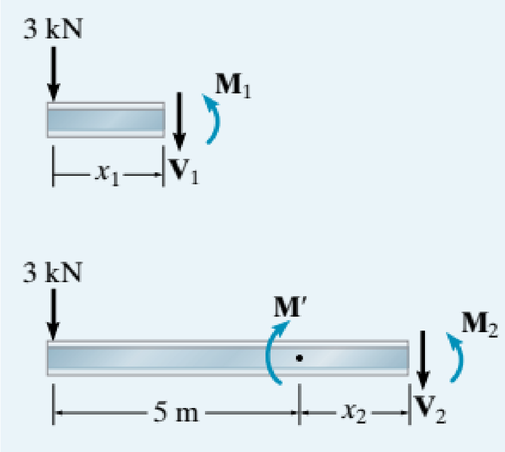
- Determine the slope at point B of the beam shown in the figure. Take $E = 200 GPa$, $I = 60(10^6) mm^4$.



Solution 4

- Since the slope at point B is to be determined, an external couple *M'* is placed on the beam at this point. Two coordinates, x_1 and x_2, must be used to determine the internal moments within the beam since there is a discontinuity, M', at B.





$$\circlearrowleft + \Sigma M = 0; \qquad M_1 + 3x_1 = 0$$

$$M_1 = -3x_1$$

$$\frac{\partial M_1}{\partial M'} = 0$$

For x_2:

$$\circlearrowleft + \Sigma M = 0; \qquad M_2 - M' + 3(5 + x_2) = 0$$

$$M_2 = M' - 3(5 + x_2)$$

$$\frac{\partial M_2}{\partial M'} = 1$$

Castigliano's Theorem. Setting $M' = 0$, its actual value, and applying Eq. 9–29, we have

$$\theta_B = \int_0^L M\left(\frac{\partial M}{\partial M'}\right)\frac{dx}{EI}$$

$$= \int_0^5 \frac{(-3x_1)(0)\,dx_1}{EI} + \int_0^5 \frac{-3(5 + x_2)(1)\,dx_2}{EI} = -\frac{112.5 \text{ kN}\cdot\text{m}^2}{EI}$$

or

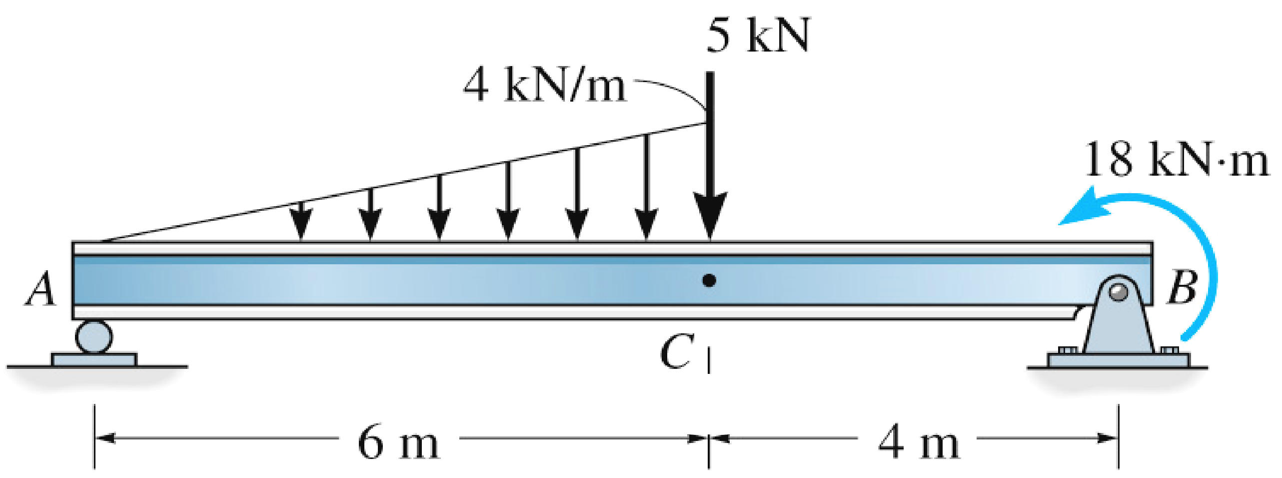
$$\theta_B = \frac{-112.5 \text{ kN}\cdot\text{m}^2}{200(10^6) \text{ kN/m}^2\left[60(10^6) \text{ mm}^4\right](10^{-12} \text{ m}^4/\text{mm}^4)}$$

$$= -0.00938 \text{ rad} \qquad \textit{Ans.}$$

The negative sign indicates that θ_B is opposite to the direction of the couple moment **M'**. Note the similarity between this solution and that

Example 5

Determine the vertical displacement of point C of the steel beam shown in Fig. 14–44a. Take $E_{st} = 200$ GPa, $I = 125(10^{-6})$ m^4.

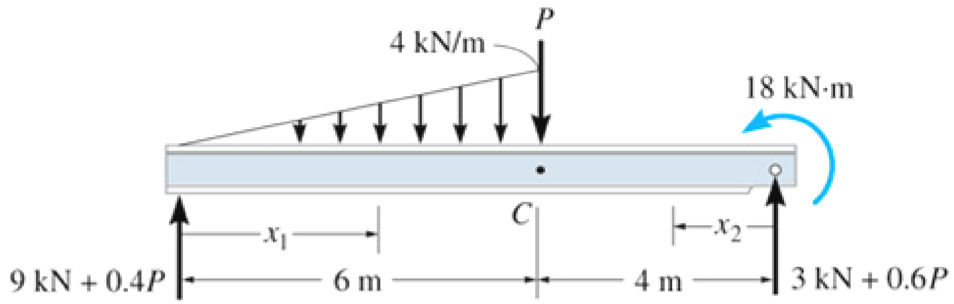


(a)

SOLUTION

External Force **P.** A vertical force **P** is applied at point C, Fig. 14–44b. Later this force will be set equal to the fixed value of 5 kN.

(b)



Solution 5

Internal Moments M. In this case two x coordinates are needed for the integration since the load is discontinuous at C. Using the method of sections, Fig. 14–44c, the internal moments and partial derivatives are determined as follows:

For x_1,

$$+\Sigma M_{NA} = 0; \quad M_1 + \frac{1}{3}x_1^2\left(\frac{x_1}{3}\right) - (9 + 0.4P)x_1 = 0$$

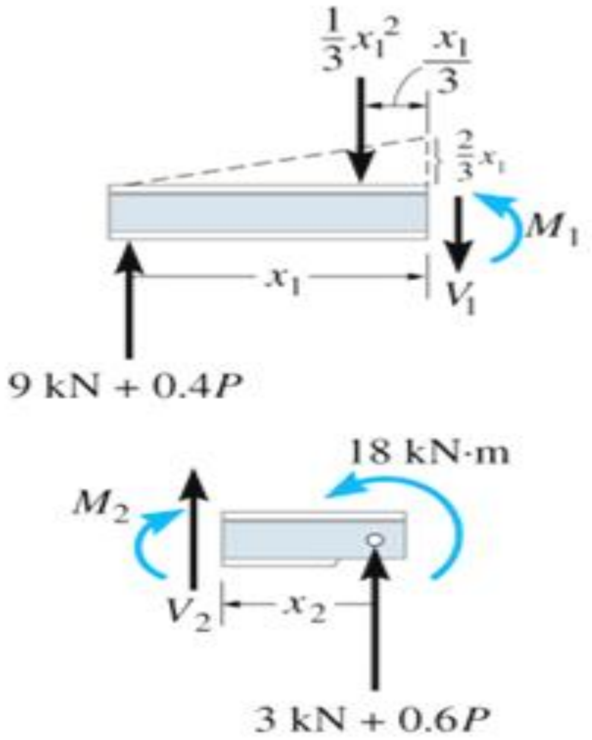
$$M_1 = (9 + 0.4P)x_1 - \frac{1}{9}x_1^3$$

$$\frac{\partial M_1}{\partial P} = 0.4x_1$$

For x_2,

$$\circlearrowleft + \Sigma M_{NA} = 0; \quad -M_2 + 18 + (3 + 0.6P)x_2 = 0$$

$$M_2 = 18 + (3 + 0.6P)x_2$$



(c)

Solution 5

$$\frac{\partial M_2}{\partial P} = 0.6x_2$$

Castigliano's Second Theorem. Setting $P = 5$ kN and applying Eq. 14–49, we have

$$\Delta_{C_v} = \int_0^L M\left(\frac{\partial M}{\partial P}\right)\frac{dx}{EI}$$

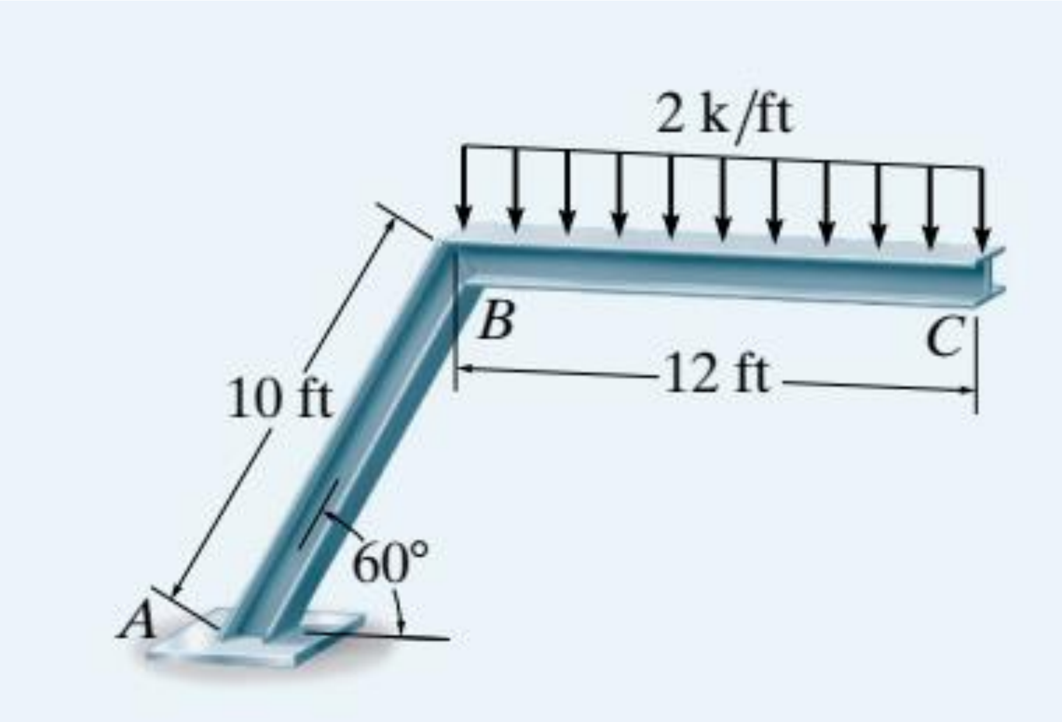
$$= \int_0^6 \frac{(11x_1 - \frac{1}{9}x_1^3)(0.4x_1)\,dx_1}{EI} + \int_0^4 \frac{(18 + 6x_2)(0.6x_2)\,dx_2}{EI}$$

$$= \frac{410.9 \text{ kN}\cdot\text{m}^3}{[200(10^6)\text{ kN/m}^2]\,125(10^{-6})\text{ m}^4}$$

$$= 0.0164 \text{ m} = 16.4 \text{ mm}$$ ***Ans.***

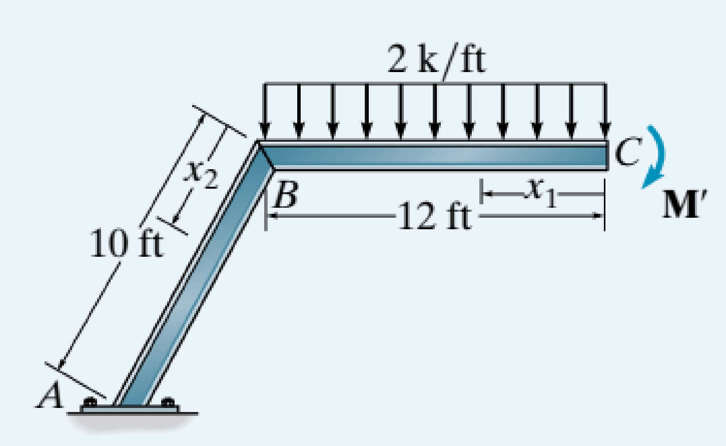
Example 6

- Determine the slope at point C of the two-member frame shown in the Figure below. The support at A is fixed. Take $E = 29000\ ksi$, $I = 600in^4$.



Solution 6

- A variable moment M’ is applied to the fra-me at point C, since the slope at this point is to be determined. Due to the discontinuity of internal loading at B, two coordinates, x_1 and x_2, are chosen



For x_1:

$$\circlearrowleft +\Sigma M = 0; \qquad -M_1 - 2x_1\left(\frac{x_1}{2}\right) - M' = 0$$

$$M_1 = -\left(x_1^2 + M'\right)$$

$$\frac{\partial M_1}{\partial M'} = -1$$

For x_2:

$$\circlearrowleft +\Sigma M = 0; \qquad -M_2 - 24(x_2 \cos 60^\circ + 6) - M' = 0$$

$$M_2 = -24(x_2 \cos 60^\circ + 6) - M'$$

$$\frac{\partial M_2}{\partial M'} = -1$$

Solution 6

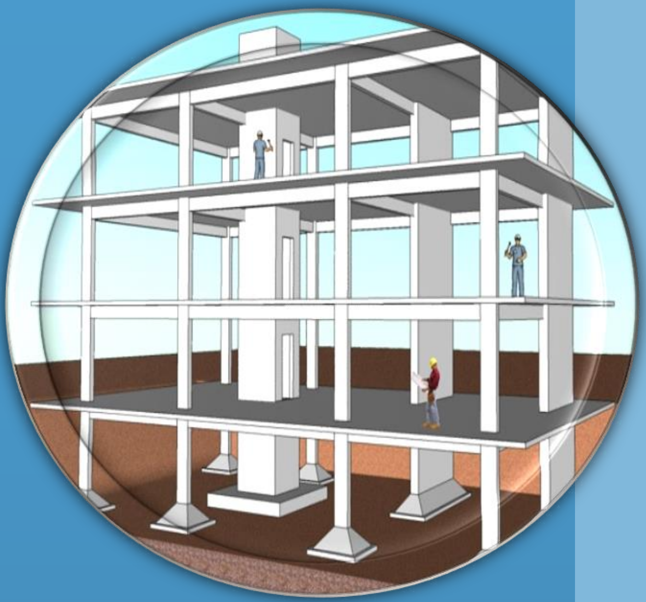
Castigliano's Theorem. Setting $M' = 0$ and applying Eq. 9–29 yields

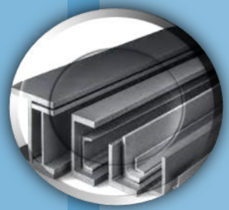
$$\theta_C = \int_0^L M\left(\frac{\partial M}{\partial M'}\right)\frac{dx}{EI}$$

$$= \int_0^{12} \frac{(-x_1^2)(-1)\,dx_1}{EI} + \int_0^{10} \frac{-24(x_2 \cos 60^\circ + 6)(-1)\,dx_2}{EI}$$

$$= \frac{576\text{ k}\cdot\text{ft}^2}{EI} + \frac{2040\text{ k}\cdot\text{ft}^2}{EI} = \frac{2616\text{ k}\cdot\text{ft}^2}{EI}$$

$$\theta_C = \frac{2616\text{ k}\cdot\text{ft}^2(144\text{ in}^2/\text{ft}^2)}{29(10^3)\text{ k/in}^2(600\text{ in}^4)} = 0.0216\text{ rad}$$ *Ans.*





Thank You!