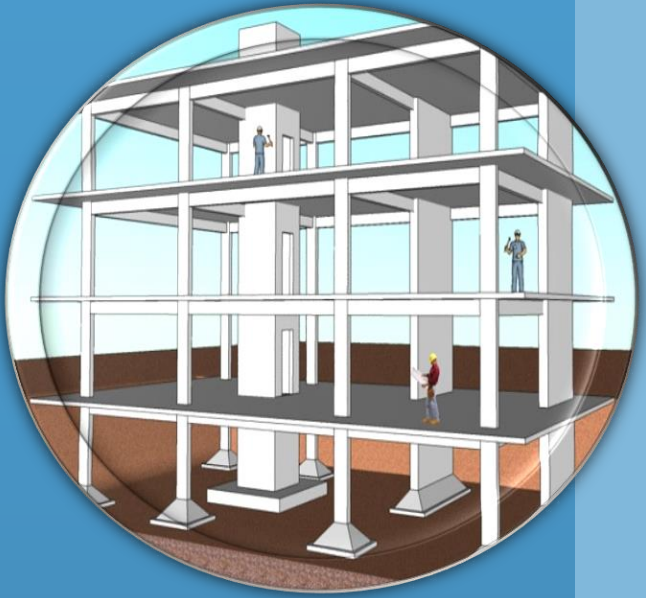
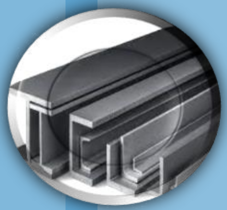
UNIVERSITY OF ZAMBIA
School of Engineering
Department of Civil & Environmental Eng.



CEE 3222: THEORY OF STRUCTURES

Lecture 4.2

DISPLACEMENT METHODS FOR SIS – SLOPE-DEFLECTION METHOD



Contents

Introduction

- For a coplanar structure there are at most three equilibrium equations for each part, so that if there is a total of n parts and r force and moment reaction components, we have

$$r = 3n, \text{ statically determinate}$$
$$r > 3n, \text{ statically indeterminate}$$

- This indeterminacy may arise as a result of:-
  - Added supports
  - Added members
  - General form of the structure (e.g. RC)

Introduction

Advantages

- The maximum stress and deflection of an indeterminate structure are generally smaller than those of its statically determinate
- Tendency to redistribute its load to its redundant supports in cases where faulty design or overloading occurs
- statically indeterminate structures can support a loading with thinner members and with increased stability compared to their statically determinate counterparts

Disadvantages

- Costly to fabricate joints for statically indeterminate.
- Because redundant support reactions, Great care to prevent differential displacement of the supports.
- any deformation, such as that caused by relative support displacement, or changes in member lengths caused by temperature or fabrication errors, will introduce additional stresses in the structure, which must be considered when designing indeterminate structures.

Introduction

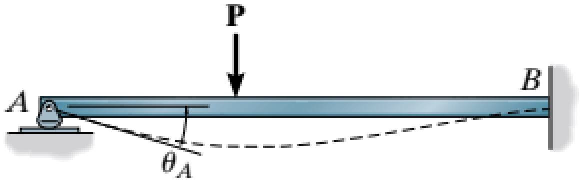
- Satisfy equilibrium, compatibility, and force-displacement
  - Equilibrium:- The reactive forces hold the structure at rest.
  - Compatibility:- structure fit together without breaks or overlaps
  - Force-displacement:- Structure carries the Load without excessive displacements
- For a statically indeterminate structure, they are the **force or flexibility method**, and **the displacement or stiffness method**.

	Unknowns	Equations Used for Solution	Coefficients of the Unknowns
Force Method	Forces	Compatibility and Force Displacement	Flexibility Coefficients
Displacement Method	Displacements	Equilibrium and Force Displacement	Stiffness Coefficients

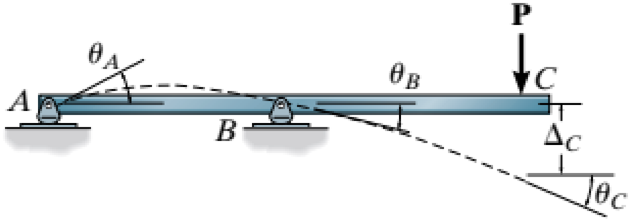
Slope Deflection Method

- All structures must satisfy equilibrium, load-displacement, and compatibility of displacements requirements in order to ensure their safety.
- The displacement method works opposite to the force method. It first requires satisfying equilibrium equations for the structure. To do this the unknown displacements are written in terms of the loads by using the load-displacement relations, then these equations are solved for the displacements.
- The compatibility equations using the load-displacement relations determine the unknowns.

Slope Deflection Method – Degree of Freedom



(a)



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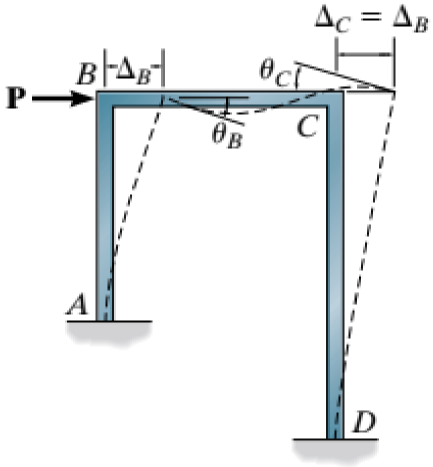


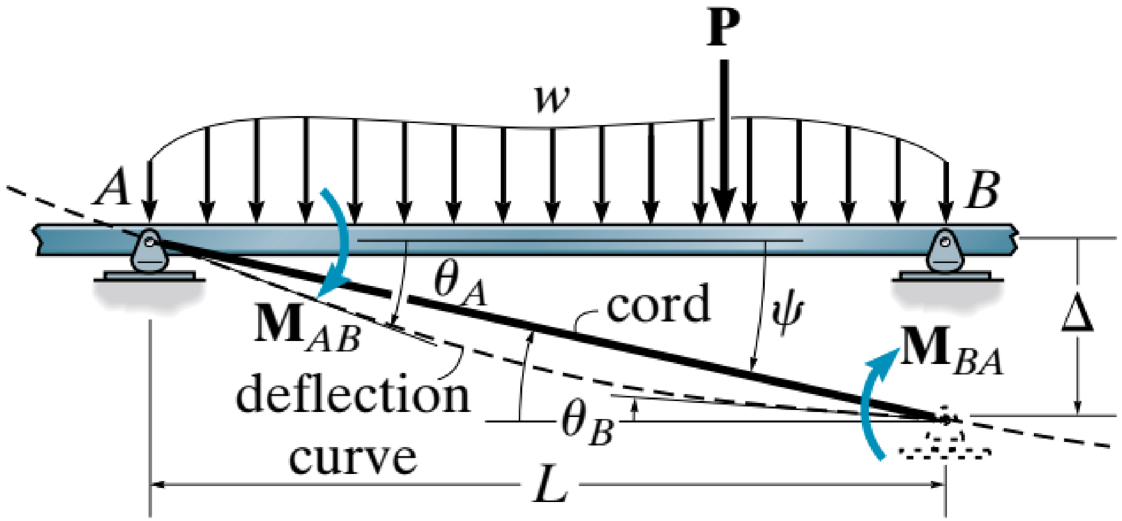
(c)

- This beam is kinematic indeterminate to the first degree, one angular displacement θ_A.
- This beam is kinematic indeterminate to the fourth degree. Thus three angular displacemebts θ_A, θ_B, θ_C and the displacement Δ_C.
- This frame is kinematic indeterminate to the three degree. Thus three angular displacements θ_B, θ_C and the equal displacement at B and C Δ_C=Δ_B.

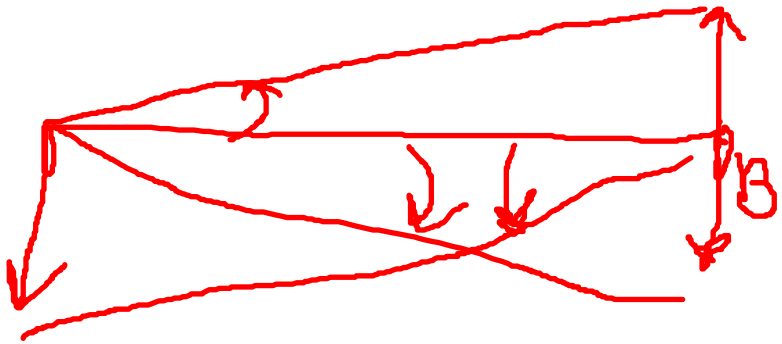
- The displacements are referred to as the degrees of freedom for the structure. They become the unknowns.
- specifying the kinematic indeterminacy or the number of unconstrained degrees of freedom for the structure is a necessary first step when applying a displacement method of analysis.
- It identifies the number of unknowns

Slope Deflection Method – THE EQUATION

- The slope deflection method is not as involving as the force methods especially with higher degrees of indeterminate. It requires less work
- Consider a span AB of a continuous beam, loaded with some loads and a constant EI. We need to relate M_{AB} and M_{BA} (internal end moments) to the degrees of freedom θ_A and θ_B and linear displacement Δ_B.

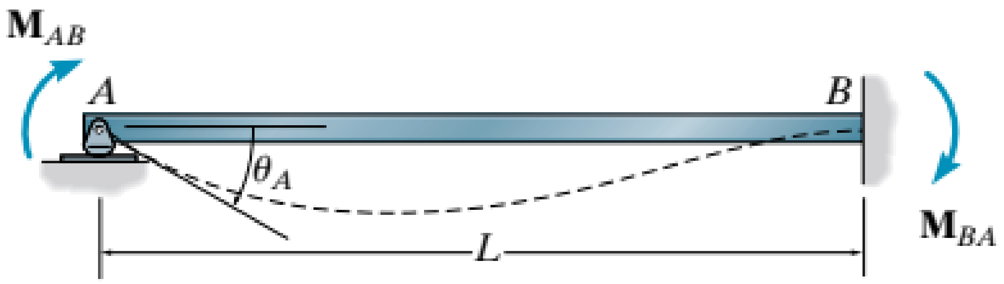


EI is constant
positive sign convention



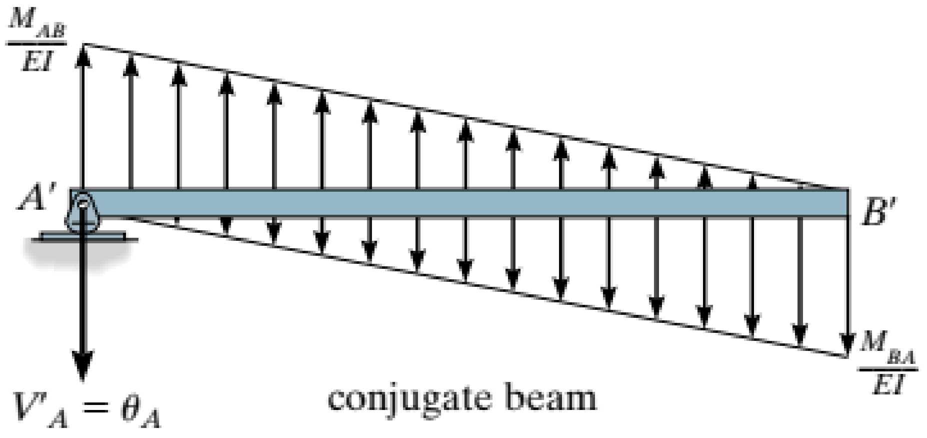
- This beam is kinematic indeterminate to the Third degree. Thus three rotations θ_A, θ_B, and the displacement Δ_B.

Slope Deflection Method – Angular Displacement at A, θ_A



real beam

(a)



conjugate beam

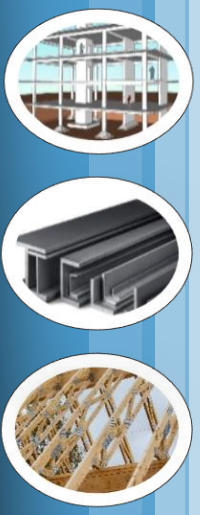
(b)

$$\circlearrowleft +\Sigma M_{A'} = 0; \qquad \left[\frac{1}{2}\left(\frac{M_{AB}}{EI}\right)L\right]\frac{L}{3} - \left[\frac{1}{2}\left(\frac{M_{BA}}{EI}\right)L\right]\frac{2L}{3} = 0$$

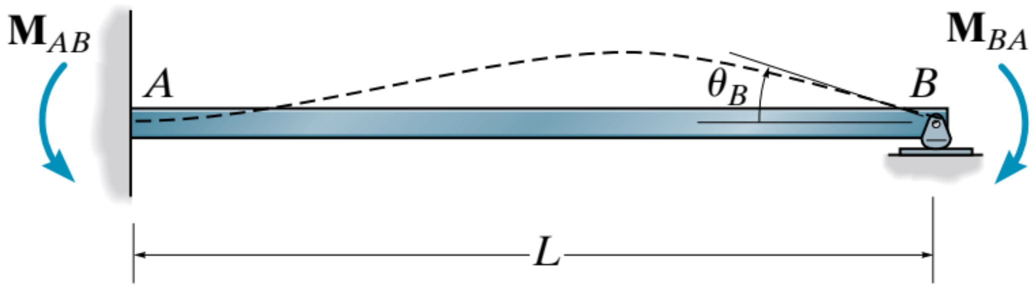
$$\circlearrowleft +\Sigma M_{B'} = 0; \qquad \left[\frac{1}{2}\left(\frac{M_{BA}}{EI}\right)L\right]\frac{L}{3} - \left[\frac{1}{2}\left(\frac{M_{AB}}{EI}\right)L\right]\frac{2L}{3} + \theta_A L = 0$$

$$\boxed{M_{AB} = \frac{4EI}{L}\theta_A}$$

$$\boxed{M_{BA} = \frac{2EI}{L}\theta_A}$$



Angular Displacement at B, θ_B



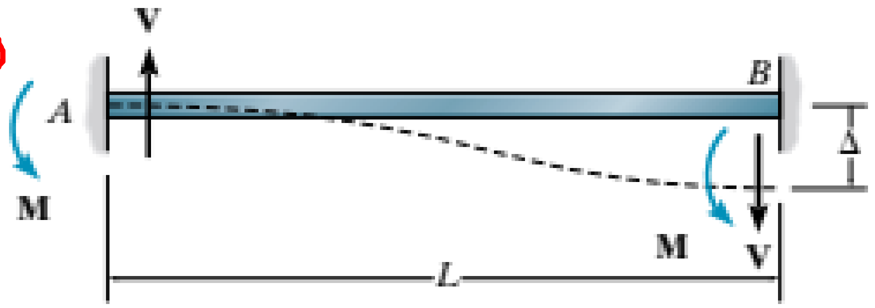
$$M_{BA} = \frac{4EI}{L}\theta_B$$

$$M_{AB} = \frac{2EI}{L}\theta_B$$

Linear Displacement at B, Δ_B

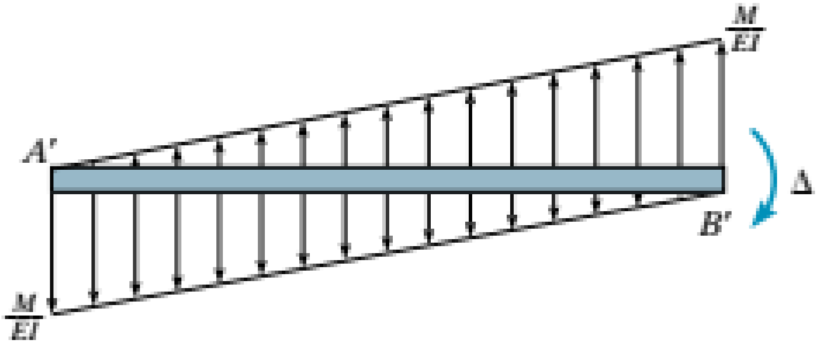
$$\zeta + \Sigma M_{B'} = 0; \quad \left[\frac{1}{2}\frac{M}{EI}(L)\left(\frac{2}{3}L\right)\right] - \left[\frac{1}{2}\frac{M}{EI}(L)\left(\frac{1}{3}L\right)\right] - \Delta = 0$$

$$M_{AB} = M_{BA} = M = \frac{-6EI}{L^2}\Delta$$



real beam

(a)

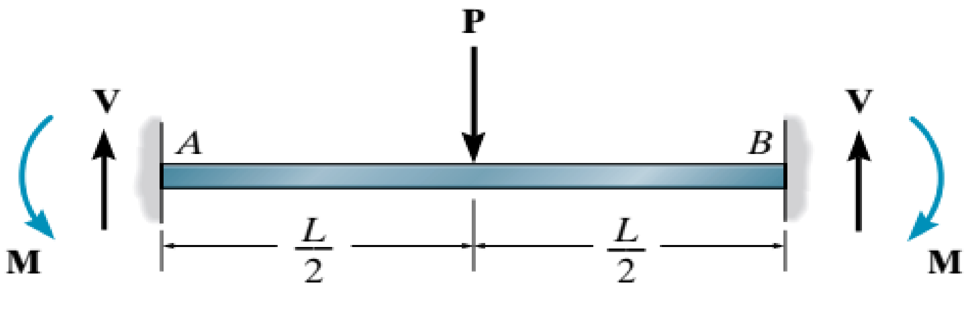


conjugate beam

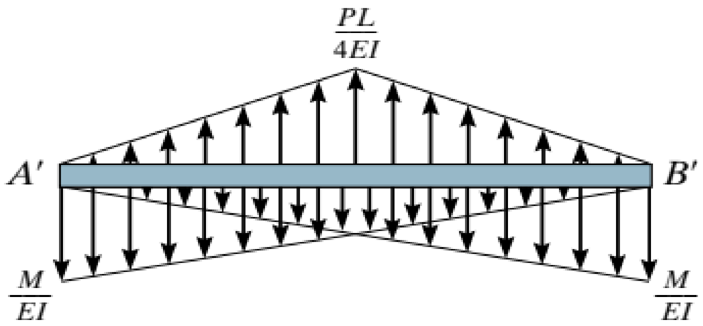
(b)

Slope Deflection Method – THE EQUATIONS

- FIXED END MOMENT:- In general, however, the linear or angular displacements of the nodes are caused by loadings acting on the span of the member, not by moments acting at its nodes.
- In order to develop the slope-deflection equations, we must transform these span loadings into equivalent moments acting at the nodes and then use the load-displacement relationships just derived.



real beam
(a)



conjugate beam
(b)

- Since we require the slope at each end to be zero

$$+\uparrow\Sigma F_y = 0; \quad \left[\frac{1}{2}\left(\frac{PL}{4EI}\right)L\right] - 2\left[\frac{1}{2}\left(\frac{M}{EI}\right)L\right] = 0$$

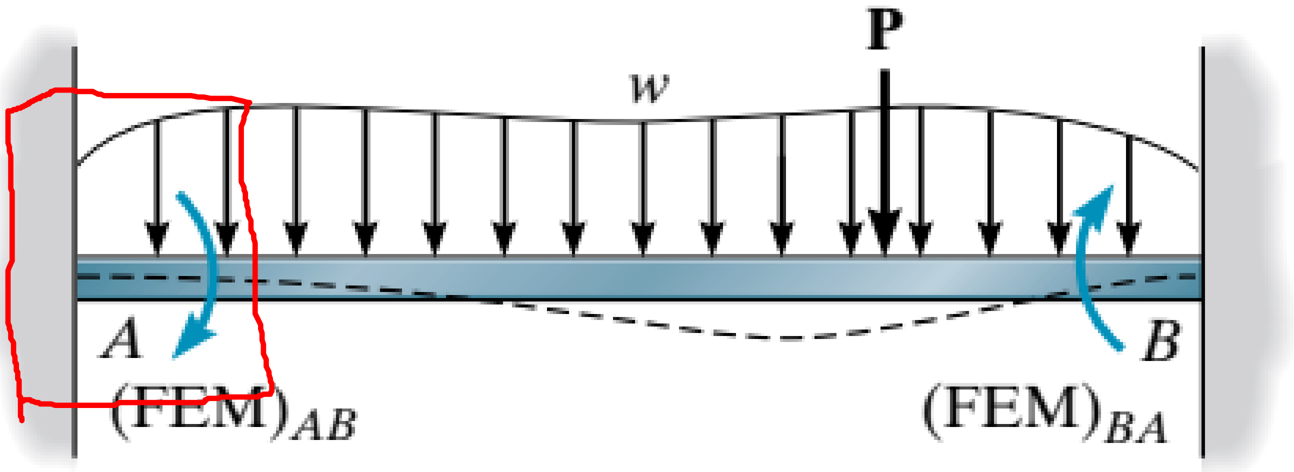
- This moment is called fixed end moment, +ve at A and –ve at B

$$M = \frac{PL}{8}$$

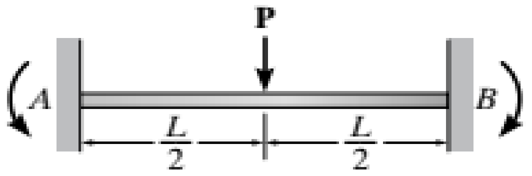
Slope Deflection Method – Fixed End Moments

- For convenience in solving problems, fixed-end moments have been calculated for other loadings and are tabulated on the inside back cover of the book. Assuming these FEMs have been determined for a specific problem, we have

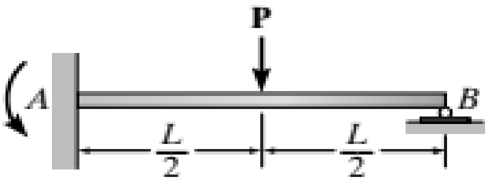
$$M_{AB} = (\mathrm{FEM})_{AB} \qquad M_{BA} = (\mathrm{FEM})_{BA}$$



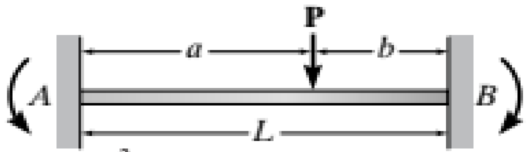
Fixed end moments



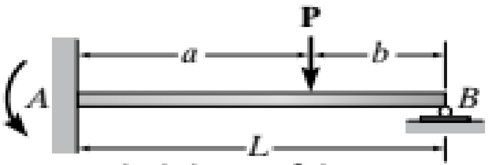
$(FEM)_{AB} = \frac{PL}{8}$ $(FEM)_{BA} = \frac{PL}{8}$



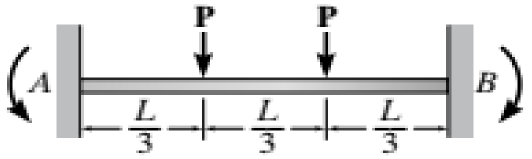
$(FEM)'_{AB} = \frac{3PL}{16}$



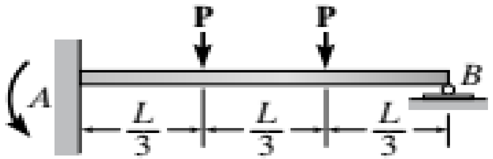
$(FEM)_{AB} = \frac{Pb^2a}{L^2}$ $(FEM)_{BA} = \frac{Pa^2b}{L^2}$



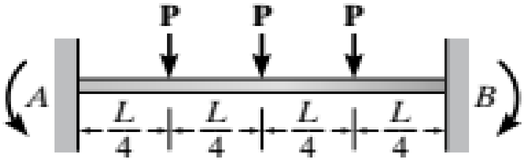
$(FEM)'_{AB} = \left(\frac{P}{L^2}\right)\left(b^2a + \frac{a^2b}{2}\right)$



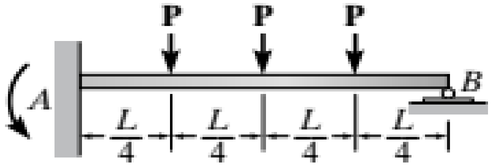
$(FEM)_{AB} = \frac{2PL}{9}$ $(FEM)_{BA} = \frac{2PL}{9}$



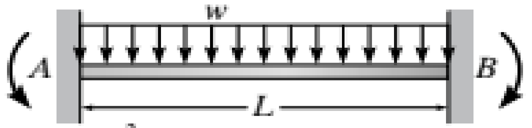
$(FEM)'_{AB} = \frac{PL}{3}$



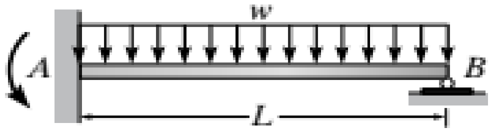
$(FEM)_{AB} = \frac{5PL}{16}$ $(FEM)_{BA} = \frac{5PL}{16}$



$(FEM)'_{AB} = \frac{15PL}{32}$

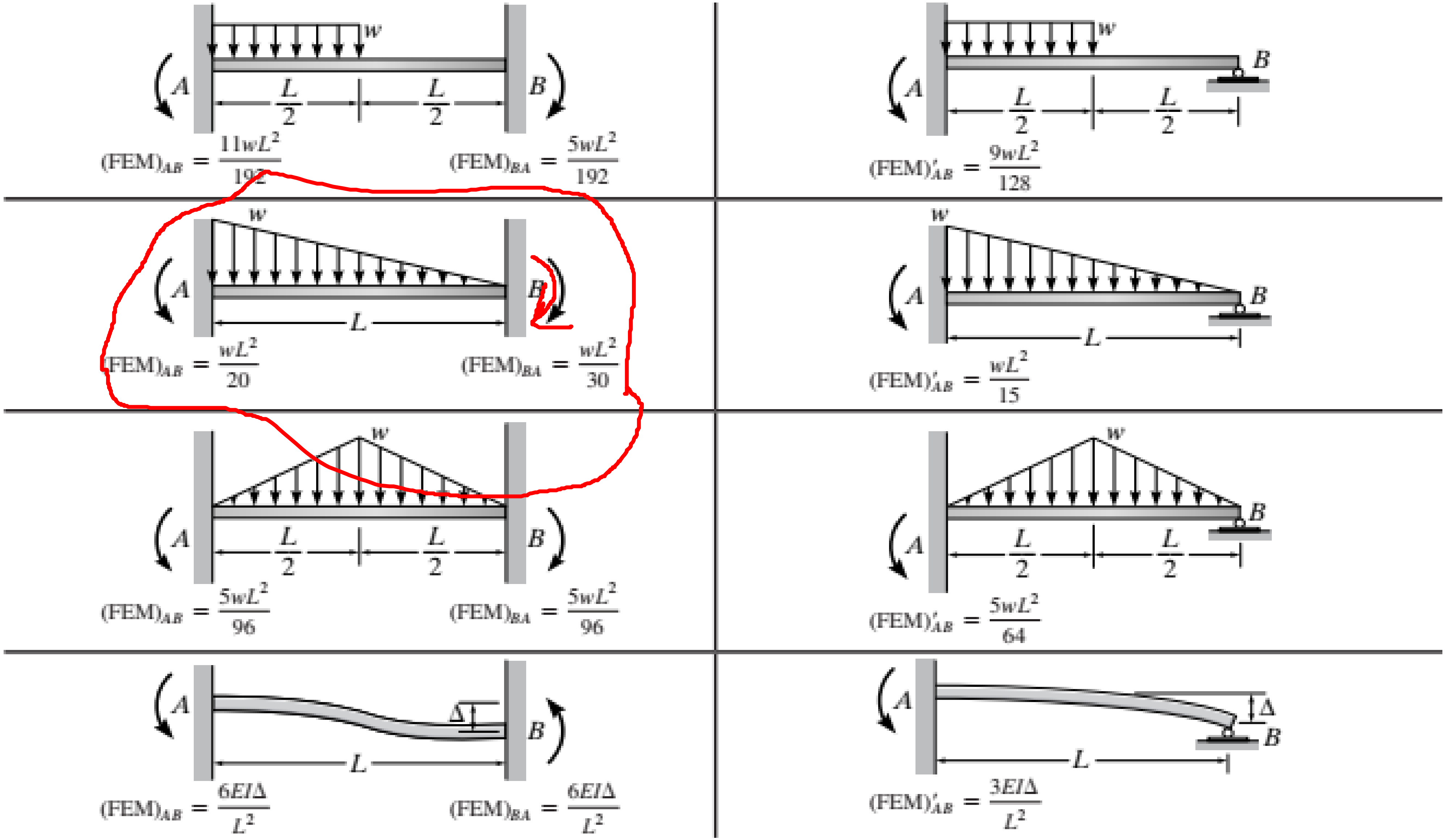


$(FEM)_{AB} = \frac{wL^2}{12}$ $(FEM)_{BA} = \frac{wL^2}{12}$



$(FEM)'_{AB} = \frac{wL^2}{8}$

Fixed end moments



Slope Deflection Method – THE EQUATIONS

- By Superposition, we will add the Moments together

$$M_{AB} = 2E\left(\frac{I}{L}\right)\left[2\theta_A + \theta_B - 3\left(\frac{\Delta}{L}\right)\right] + (\text{FEM})_{AB}$$

$$M_{BA} = 2E\left(\frac{I}{L}\right)\left[2\theta_B + \theta_A - 3\left(\frac{\Delta}{L}\right)\right] + (\text{FEM})_{BA}$$

member stiffness $k = I/L$

span's cord rotation as ψ (psi) $= \Delta/L$

- We combine the above 2 equations. Referring to one end of the span as the near end (N) and the other end as the far end (F). We get the general **slope-deflection equation.**

$$M_N = 2Ek(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

For Internal Span or End Span with Far End Fixed

where

M_N = internal moment in the near end of the span; this moment is *positive clockwise* when acting on the span.

E, k = modulus of elasticity of material and span stiffness $k = I/L$.

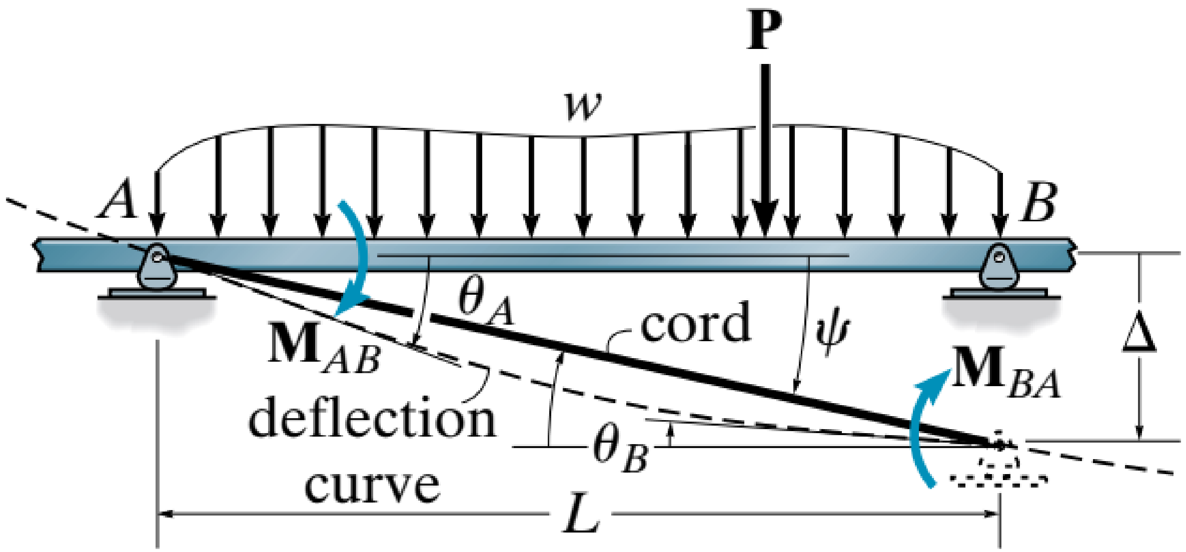
θ_N, θ_F = near- and far-end slopes or angular displacements of the span at the supports; the angles are measured in *radians* and are *positive clockwise*.

ψ = span rotation of its cord due to a linear displacement, that is, $\psi = \Delta/L$; this angle is measured in *radians* and is *positive clockwise*.

$(\text{FEM})_N$ = fixed-end moment at the near-end support; the moment is *positive clockwise* when acting on the span; refer to the table on the inside back cover for various loading conditions.

Slope Deflection Method – THE EQUATIONS

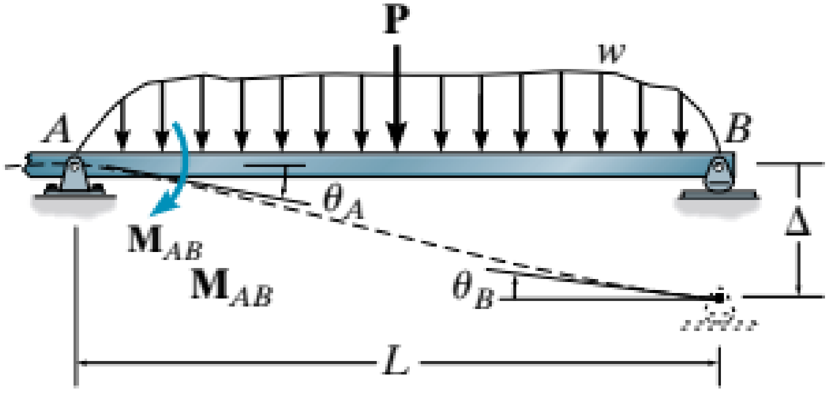
- The general slope-deflection equation when used for the solution of problems, is applied twice for each member span (AB); that is, application is from A to B and from B to A for span AB in below.



EI is constant
positive sign convention

Slope Deflection Method – THE EQUATIONS

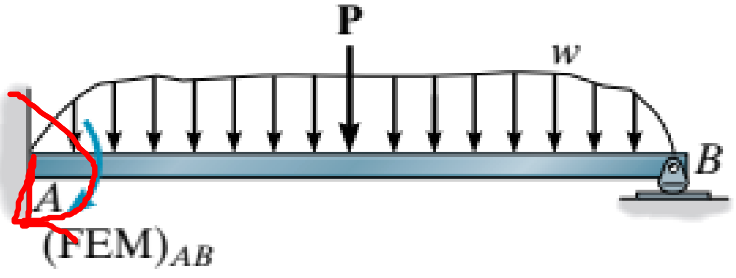
- When the end support is a pin or a roller the moment at the roller or pin must be zero;
- The angular displacement θ_B at this support does not have to be determined,



(a)

$$2\times \quad M_N = 2Ek(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N \quad \times 2$$

$$0 = 2Ek(2\theta_F + \theta_N - 3\psi) + 0$$



(b)

$$M_N = 3Ek(\theta_N - \psi) + (\text{FEM})_N$$

Only for End Span with Far End Pinned or Roller Supported

Slope Deflection Method – Beam analysis procedure

Procedure for Analysis

Degrees of Freedom

Label all the supports and joints (nodes) in order to identify the spans of the beam or frame between the nodes. By drawing the deflected shape of the structure, it will be possible to identify the number of degrees of freedom. Here each node can possibly have an angular displacement and a linear displacement. *Compatibility* at the nodes is maintained provided the members that are fixed connected to a node undergo the same displacements as the node. If these displacements are unknown, and in general they will be, then for convenience *assume* they act in the *positive direction* so as to cause *clockwise* rotation of a member or joint, Fig. 11–2.

Slope-Deflection Equations

The slope-deflection equations relate the unknown moments applied to the nodes to the displacements of the nodes for any span of the structure. If a load exists on the span, compute the FEMs using the table given on the inside back cover. Also, if a node has a linear displacement, Δ, compute $\psi = \Delta/L$ for the adjacent spans. Apply Eq. 11–8 to each end of the span, thereby generating *two* slope-deflection equations for each span. However, if a span at the *end* of a continuous beam or frame is pin supported, apply Eq. 11–10 only to the restrained end, thereby generating *one* slope-deflection equation for the span.

Equilibrium Equations

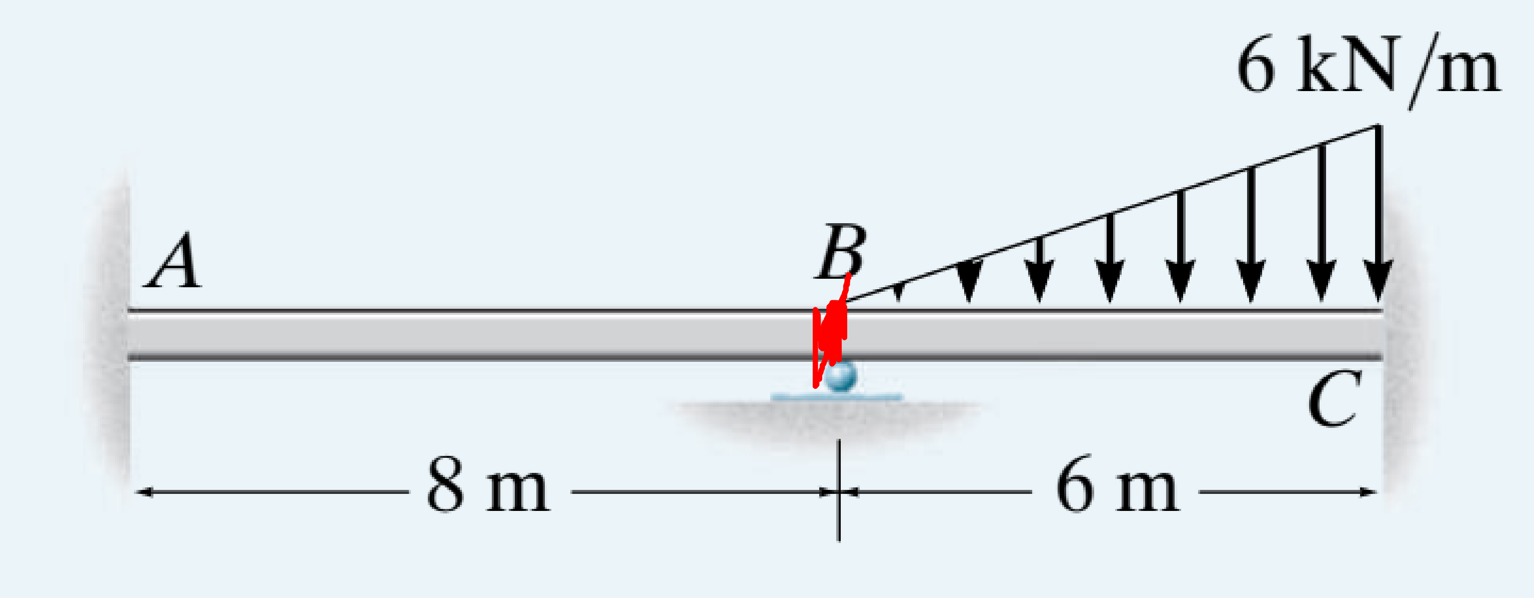
Write an equilibrium equation for each unknown degree of freedom for the structure. Each of these equations should be expressed in terms of unknown internal moments as specified by the slope-deflection equations. For beams and frames write the moment equation of equilibrium at each support, and for frames also write joint moment equations of equilibrium. If the frame sidesways or deflects horizontally, column shears should be related to the moments at the ends of the column. This is discussed in Sec. 11.5.

Substitute the slope-deflection equations into the equilibrium equations and solve for the unknown joint displacements. These results are then substituted into the slope-deflection equations to determine the internal moments at the ends of each member. If any of the results are *negative*, they indicate *counterclockwise* rotation; whereas *positive* moments and displacements are applied *clockwise*.

Example 1

- Draw the shear and moment diagrams for the beam shown in Figure below. EI is constant

4 degree



Solution 1

- Two spans considered and since no span has a roller or pin at the end, the equations will be applied twice.
- Using the FEM formulae provided, we solve

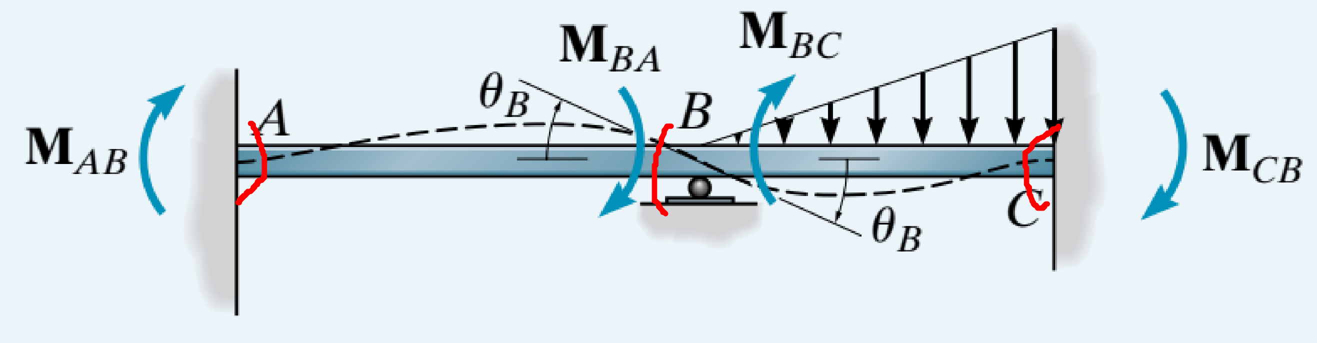
$$(\text{FEM})_{BC} = -\frac{wL^2}{30} = -\frac{6(6)^2}{30} = -7.2\ \text{kN}\cdot\text{m}$$

$$(\text{FEM})_{CB} = \frac{wL^2}{20} = \frac{6(6)^2}{20} = 10.8\ \text{kN}\cdot\text{m}$$

Note that $(\text{FEM})_{BC}$ is negative since it acts counterclockwise *on the beam* at B. Also, $(\text{FEM})_{AB} = (\text{FEM})_{BA} = 0$ since there is no load on span AB.

$\theta_A = \theta_C = 0$. Since A and C are fixed

$\psi_{AB} = \psi_{BC} = 0$ the supports do not settle.



$$M_N = 2E\left(\frac{I}{L}\right)(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

$$M_{AB} = 2E\left(\frac{I}{8}\right)[2(0) + \theta_B - 3(0)] + 0 = \frac{EI}{4}\theta_B \qquad (1)$$

Now, considering B to be the near end and A to be the far end, we have

$$M_{BA} = 2E\left(\frac{I}{8}\right)[2\theta_B + 0 - 3(0)] + 0 = \frac{EI}{2}\theta_B \qquad (2)$$

In a similar manner, for span BC we have

$$M_{BC} = 2E\left(\frac{I}{6}\right)[2\theta_B + 0 - 3(0)] - 7.2 = \frac{2EI}{3}\theta_B - 7.2 \qquad (3)$$

$$M_{CB} = 2E\left(\frac{I}{6}\right)[2(0) + \theta_B - 3(0)] + 10.8 = \frac{EI}{3}\theta_B + 10.8 \qquad (4)$$

Solution 1

Equilibrium Equations. The above four equations contain five unknowns. The necessary fifth equation comes from the condition of moment equilibrium at support B. The free-body diagram of a segment of the beam at B is shown in Fig. 11–10c. Here $\mathbf{M}_{BA}$ and $\mathbf{M}_{BC}$ are assumed to act in the positive direction to be consistent with the slope-deflection equations.* The beam shears contribute negligible moment about B since the segment is of differential length. Thus,

$$\circlearrowleft +\Sigma M_B = 0; \qquad M_{BA} + M_{BC} = 0 \qquad (5)$$

To solve, substitute Eqs. (2) and (3) into Eq. (5), which yields

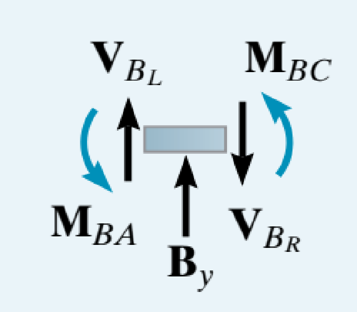
$$\theta_B = \frac{6.17}{EI}$$

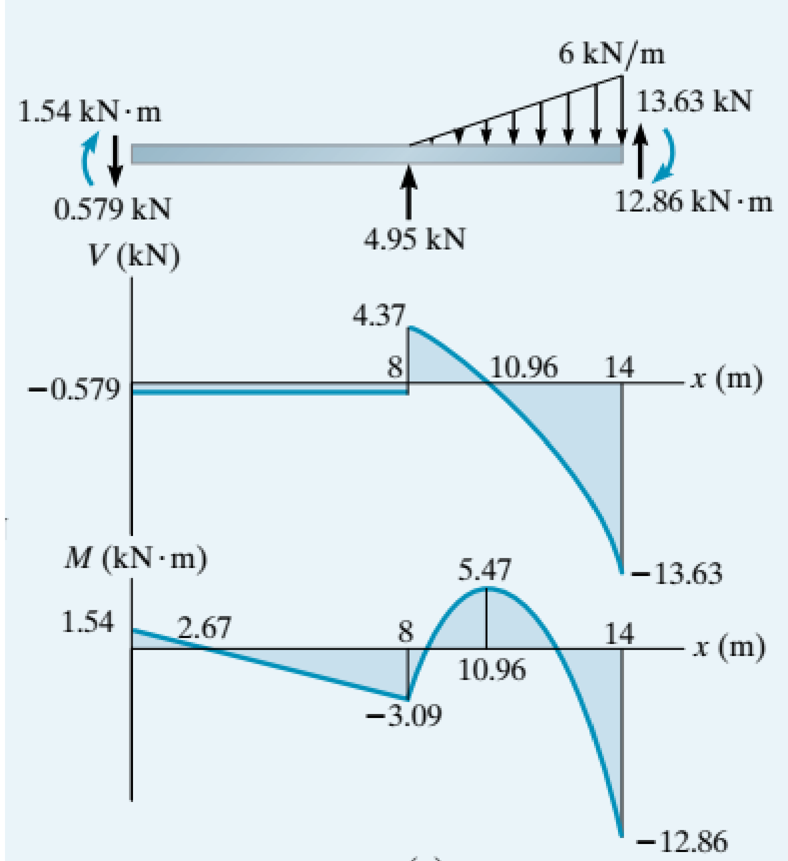
Resubstituting this value into Eqs. (1)–(4) yields

$$M_{AB} = 1.54 \text{ kN}\cdot\text{m}$$
$$M_{BA} = 3.09 \text{ kN}\cdot\text{m}$$
$$M_{BC} = -3.09 \text{ kN}\cdot\text{m}$$
$$M_{CB} = 12.86 \text{ kN}\cdot\text{m}$$

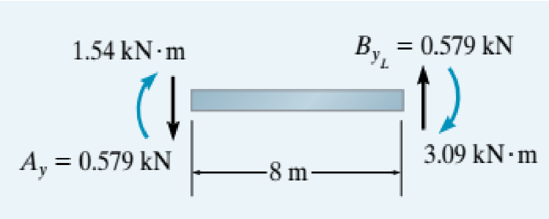
The negative value for M_{BC} indicates that this moment acts counterclockwise on the beam, not clockwise as shown in Fig. 11–10b.

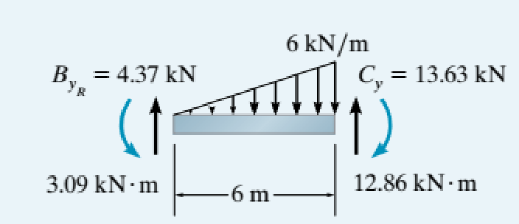
Using these results, the shears at the end spans are determined from the equilibrium equations, Fig. 11–10d. The free-body diagram of the entire beam and the shear and moment diagrams are shown in Fig. 11–10e.





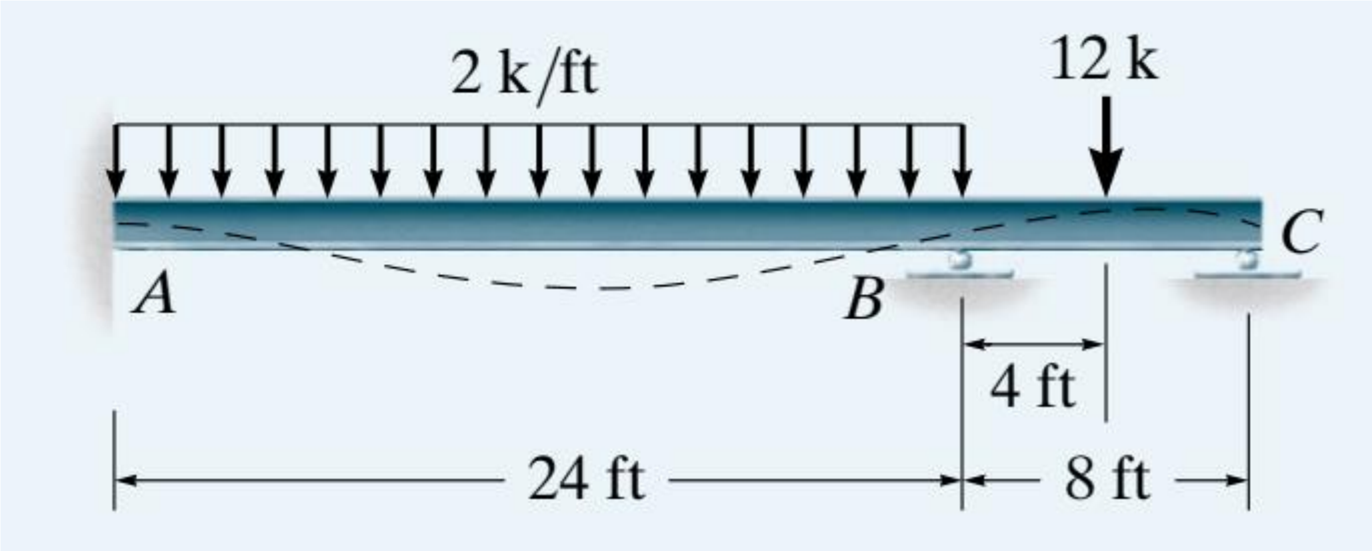
(e)





Example 2

- Draw the shear and moment diagrams for the beam shown in Figure below. EI is constant



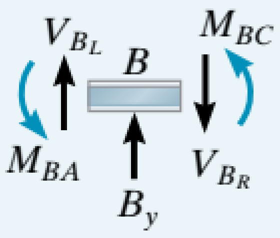
Solution 2

- Two spans considered and because of a roller at C, we apply the second equation for span BC.

$$(FEM)_{AB} = -\frac{wL^2}{12} = -\frac{1}{12}(2)(24)^2 = -96\ \text{k}\cdot\text{ft}$$

$$(FEM)_{BA} = \frac{wL^2}{12} = \frac{1}{12}(2)(24)^2 = 96\ \text{k}\cdot\text{ft}$$

$$(FEM)_{BC} = -\frac{3PL}{16} = -\frac{3(12)(8)}{16} = -18\ \text{k}\cdot\text{ft}$$



(b)

Note that $(FEM)_{AB}$ and $(FEM)_{BC}$ are negative since they act counterclockwise on the beam at A and B, respectively. Also, since the supports do not settle, $\psi_{AB} = \psi_{BC} = 0$. Applying Eq. 11–8 for span AB and realizing that $\theta_A = 0$, we have

$$M_N = 2E\left(\frac{I}{L}\right)(2\theta_N + \theta_F - 3\psi) + (FEM)_N$$

$$M_{AB} = 2E\left(\frac{I}{24}\right)[2(0) + \theta_B - 3(0)] - 96$$

$$M_{AB} = 0.08333EI\theta_B - 96 \qquad (1)$$

$$M_{BA} = 2E\left(\frac{I}{24}\right)[2\theta_B + 0 - 3(0)] + 96$$

$$M_{BA} = 0.1667EI\theta_B + 96 \qquad (2)$$

Applying Eq. 11–10 with B as the near end and C as the far end, we have

$$M_N = 3E\left(\frac{I}{L}\right)(\theta_N - \psi) + (FEM)_N$$

$$M_{BC} = 3E\left(\frac{I}{8}\right)(\theta_B - 0) - 18$$

$$M_{BC} = 0.375EI\theta_B - 18 \qquad (3)$$

Remember that Eq. 11–10 is *not* applied from C (near end) to B (far end).

Solution 2

Equilibrium Equations. The above three equations contain four unknowns. The necessary fourth equation comes from the conditions of equilibrium at the support B. The free-body diagram is shown in Fig. 11–11b. We have

$$\circlearrowleft +\Sigma M_B = 0; \qquad M_{BA} + M_{BC} = 0 \qquad (4)$$

To solve, substitute Eqs. (2) and (3) into Eq. (4), which yields

$$\theta_B = -\frac{144.0}{EI}$$

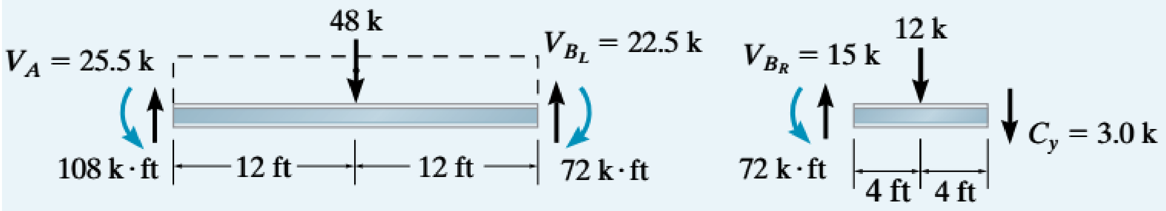
Since θ_B is negative (counterclockwise) the elastic curve for the beam has been correctly drawn in Fig. 11–11a. Substituting θ_B into Eqs. (1)–(3), we get

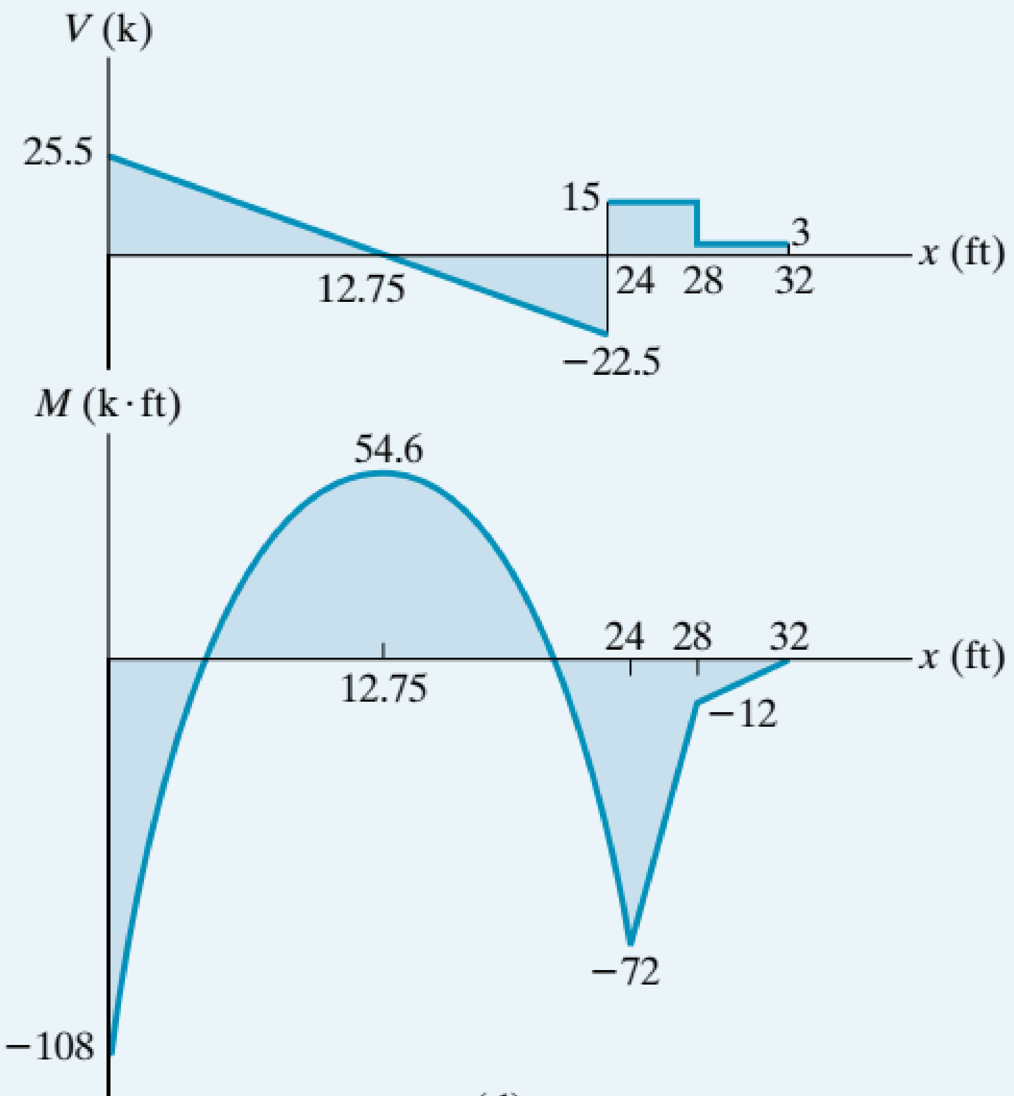
$$M_{AB} = -108.0 \text{ k}\cdot\text{ft}$$

$$M_{BA} = 72.0 \text{ k}\cdot\text{ft}$$

$$M_{BC} = -72.0 \text{ k}\cdot\text{ft}$$

Using these data for the moments, the shear reactions at the ends of the beam spans have been determined in Fig. 11–11c. The shear and moment diagrams are plotted in Fig. 11–11d.

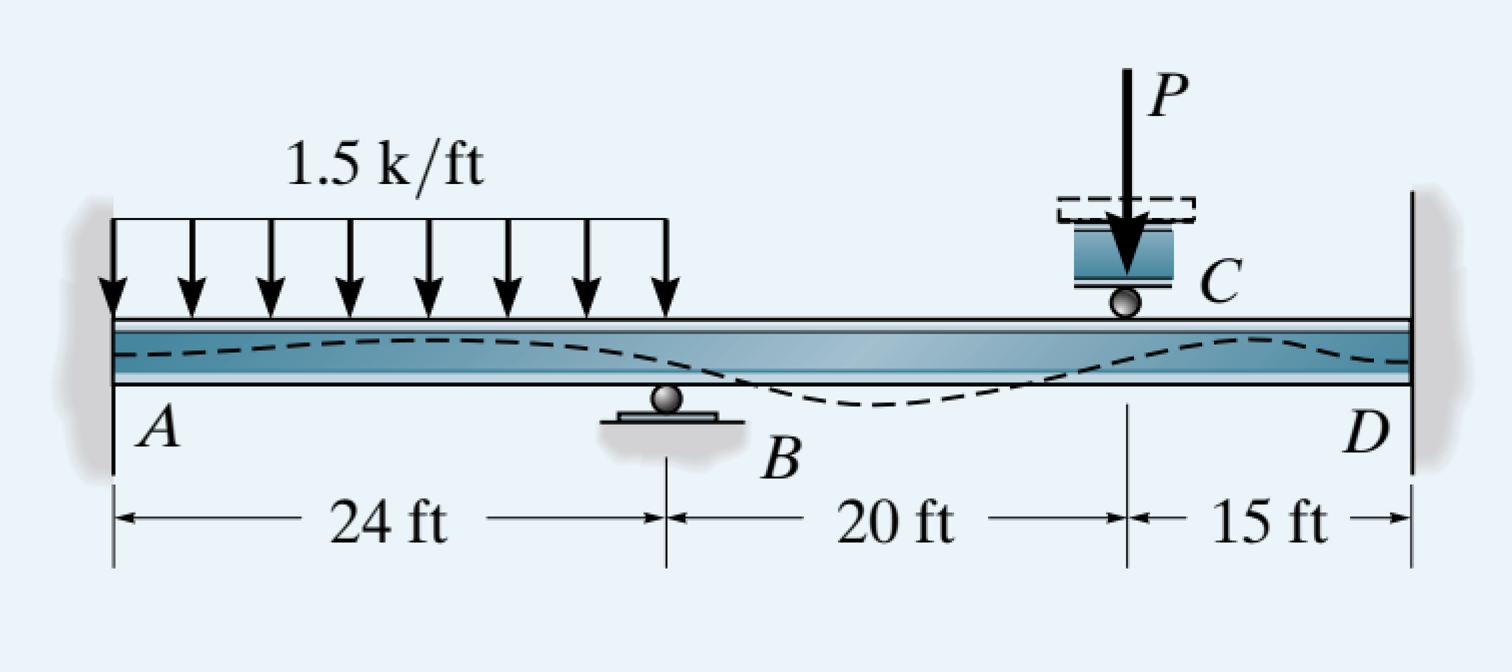




(d)

Example 3

- Determine the internal moments at the supports of the beam shown in the figure. The roller support at C is pushed downward 0.1 ft by the force P. Take $E = 29(10^3)\ ksi, I = 1500\ in^4$



Solution 3

Slope-Deflection Equations. Three spans must be considered in this problem. Equation 11–8 applies since the end supports A and D are fixed. Also, only span AB has FEMs.

$$(\text{FEM})_{AB} = -\frac{wL^2}{12} = -\frac{1}{12}(1.5)(24)^2 = -72.0 \text{ k}\cdot\text{ft}$$

$$(\text{FEM})_{BA} = \frac{wL^2}{12} = \frac{1}{12}(1.5)(24)^2 = 72.0 \text{ k}\cdot\text{ft}$$

As shown in Fig. 11–13b, the displacement (or settlement) of the support C causes ψ_{BC} to be positive, since the cord for span BC rotates clockwise, and ψ_{CD} to be negative, since the cord for span CD rotates counterclockwise. Hence,

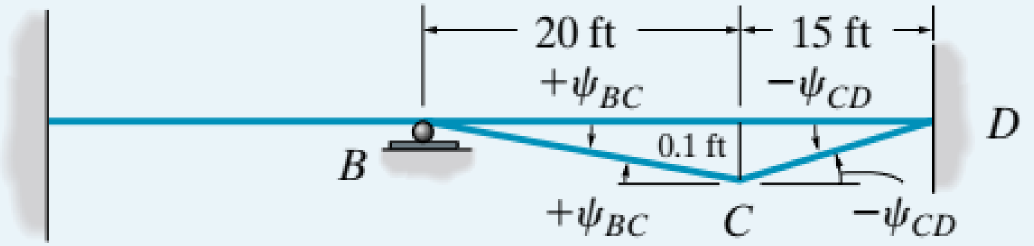
$$\psi_{BC} = \frac{0.1 \text{ ft}}{20 \text{ ft}} = 0.005 \text{ rad} \qquad \psi_{CD} = -\frac{0.1 \text{ ft}}{15 \text{ ft}} = -0.00667 \text{ rad}$$

Also, expressing the units for the stiffness in feet, we have

$$k_{AB} = \frac{1500}{24(12)^4} = 0.003014 \text{ ft}^3 \qquad k_{BC} = \frac{1500}{20(12)^4} = 0.003617 \text{ ft}^3$$

$$k_{CD} = \frac{1500}{15(12)^4} = 0.004823 \text{ ft}^3$$

Noting that $\theta_A = \theta_D = 0$ since A and D are fixed supports, and applying the slope-deflection Eq. 11–8 twice to each span, we have



For span AB:

$$M_{AB} = 2\left[29(10^3)(12)^2\right](0.003014)\left[2(0) + \theta_B - 3(0)\right] - 72$$

$$M_{AB} = 25\,173.6\theta_B - 72 \qquad (1)$$

$$M_{BA} = 2\left[29(10^3)(12)^2\right](0.003014)\left[2\theta_B + 0 - 3(0)\right] + 72$$

$$M_{BA} = 50\,347.2\theta_B + 72 \qquad (2)$$

For span BC:

$$M_{BC} = 2\left[29(10^3)(12)^2\right](0.003617)\left[2\theta_B + \theta_C - 3(0.005)\right] + 0$$

$$M_{BC} = 60\,416.7\theta_B + 30\,208.3\theta_C - 453.1 \qquad (3)$$

$$M_{CB} = 2\left[29(10^3)(12)^2\right](0.003617)\left[2\theta_C + \theta_B - 3(0.005)\right] + 0$$

$$M_{CB} = 60\,416.7\theta_C + 30\,208.3\theta_B - 453.1 \qquad (4)$$

For span CD:

$$M_{CD} = 2\left[29(10^3)(12)^2\right](0.004823)\left[2\theta_C + 0 - 3(-0.00667)\right] + 0$$

$$M_{CD} = 80\,555.6\theta_C + 0 + 805.6 \qquad (5)$$

$$M_{DC} = 2\left[29(10^3)(12)^2\right](0.004823)\left[2(0) + \theta_C - 3(-0.00667)\right] + 0$$

$$M_{DC} = 40\,277.8\theta_C + 805.6 \qquad (6)$$

Solution 3

Equilibrium Equations. These six equations contain eight unknowns. Writing the moment equilibrium equations for the supports at B and C, Fig. 10–13b, we have

$$\circlearrowleft +\Sigma M_B = 0; \qquad M_{BA} + M_{BC} = 0 \qquad (7)$$

$$\circlearrowleft +\Sigma M_C = 0; \qquad M_{CB} + M_{CD} = 0 \qquad (8)$$

In order to solve, substitute Eqs. (2) and (3) into Eq. (7), and Eqs. (4) and (5) into Eq. (8). This yields

$$\theta_C + 3.667\theta_B = 0.01262$$

$$-\theta_C - 0.2140\theta_B = 0.00250$$

Thus,

$$\theta_B = 0.00438 \text{ rad} \qquad \theta_C = -0.00344 \text{ rad}$$

The negative value for θ_C indicates counterclockwise rotation of the tangent at C, Fig. 11–13a. Substituting these values into Eqs. (1)–(6) yields

$$M_{AB} = 38.2 \text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{BA} = 292 \text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

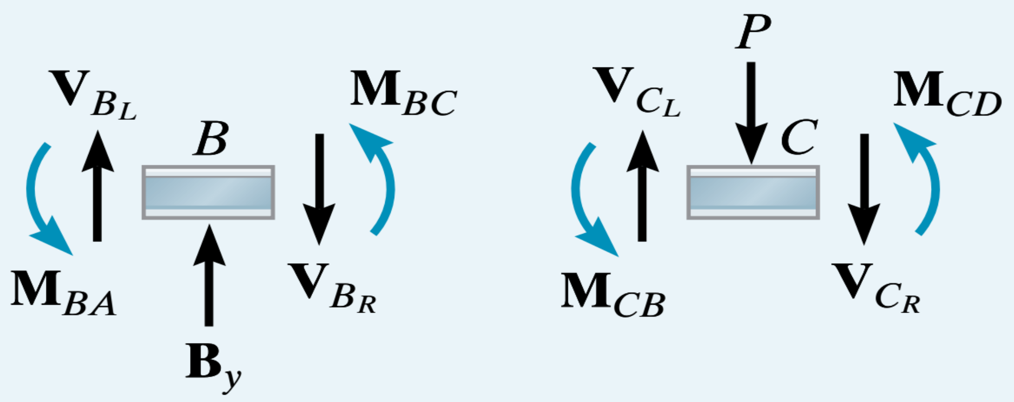
$$M_{BC} = -292 \text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{CB} = -529 \text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{CD} = 529 \text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

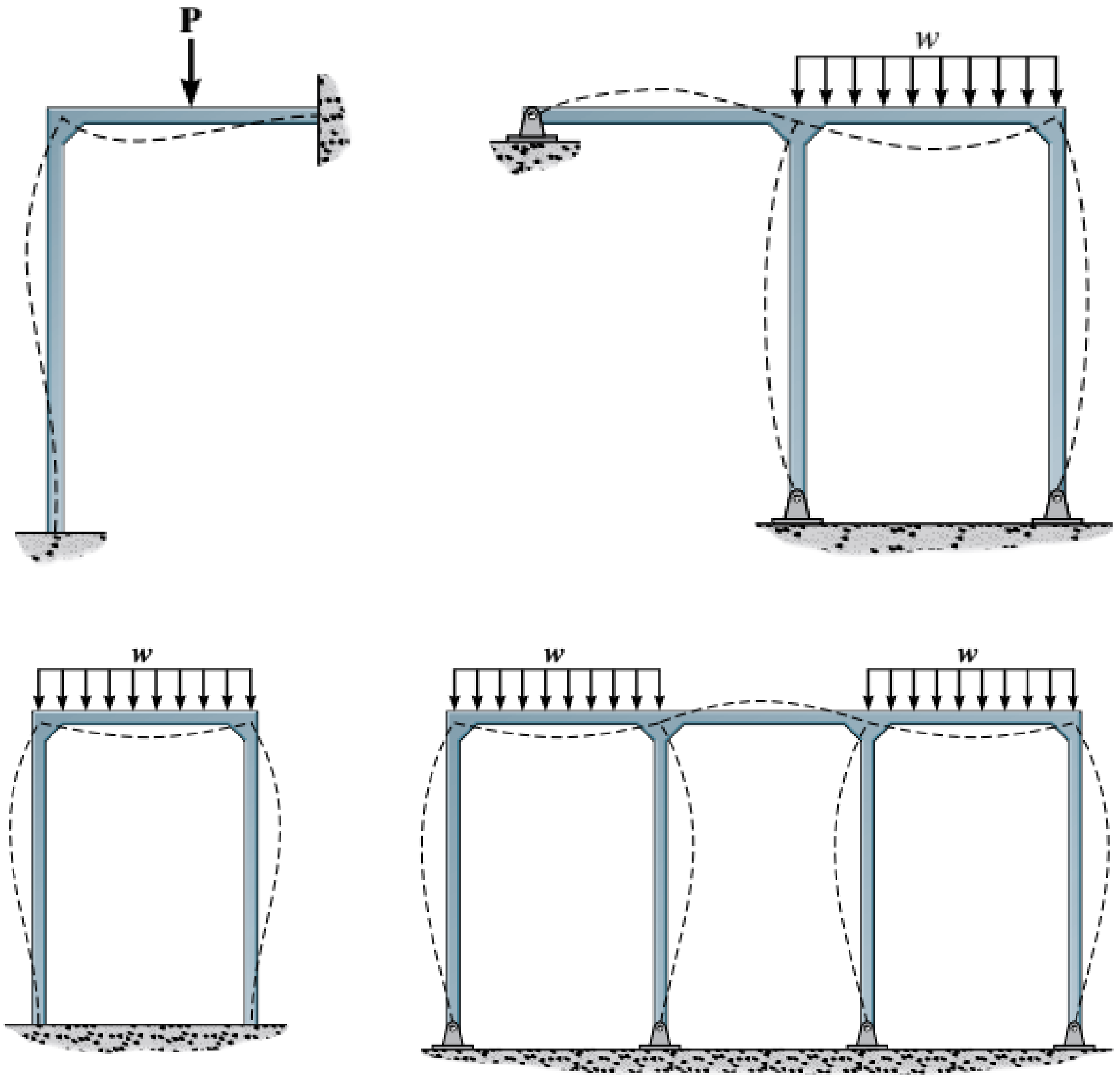
$$M_{DC} = 667 \text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

Apply these end moments to spans BC and CD and show that $V_{C_L} = 41.05$ k, $V_{C_R} = -79.73$ k and the force on the roller is $P = 121$ k.

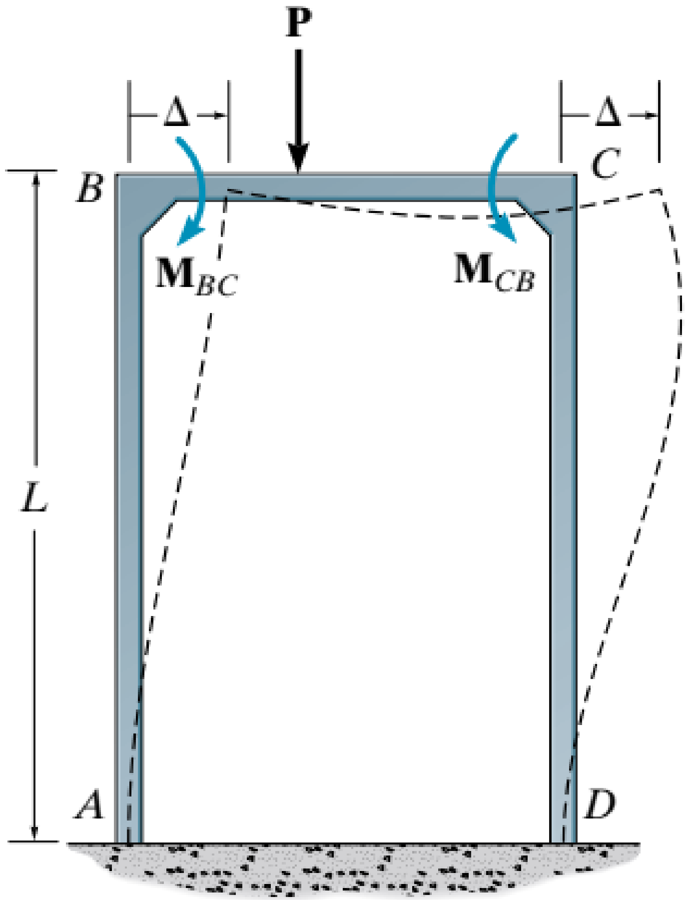


Slope Deflection Method – Frame analysis

- A frame will not sidesway, or be displaced to the left or right, provided it is properly restrained.
- Also, no sidesway will occur in an unrestrained frame provided it is symmetric with respect to both loading and geometry.
- For both cases the term $\psi = 0$, since bending does not cause the joints to have a linear displacement.
- The procedure for analysis is as outlined for beams



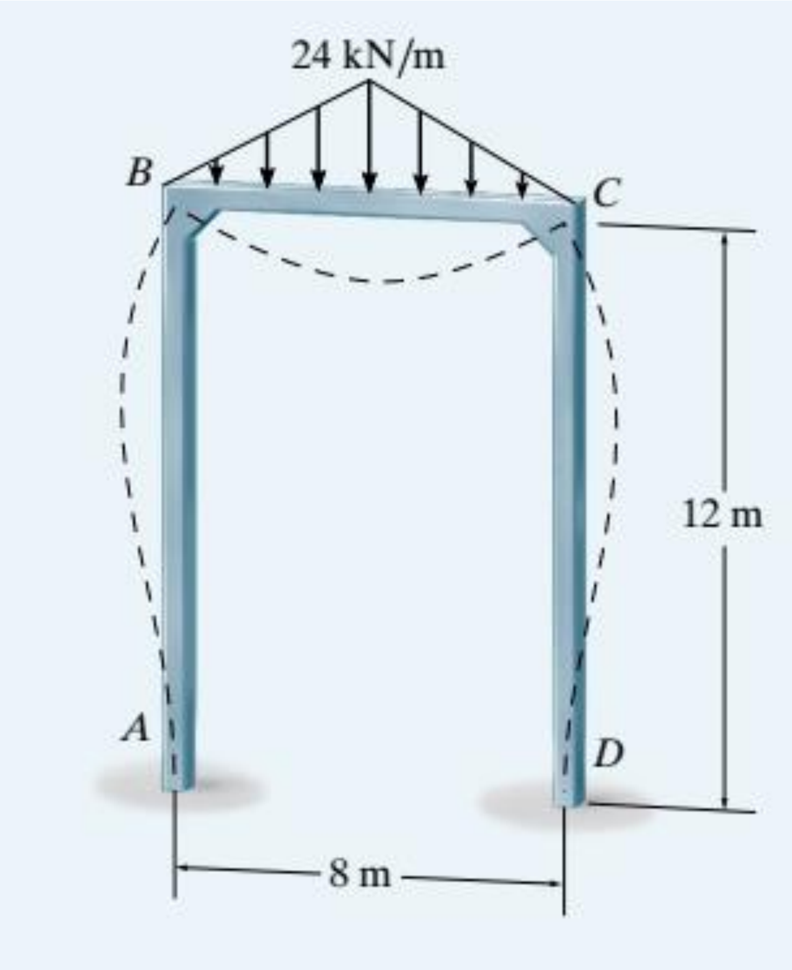
Slope Deflection Method – Frame analysis



- A frame will sidesway, or be displaced to the side, when it or the loading acting on it is non symmetric.
- Eg P causes unequal moments M_{BC} (tend to display to joint B to the right) and M_{CB} (tend to display to joint C to the left) .
- The term $\psi = \Delta/L$.
- we must write force equilibrium equations in order to obtain the complete solution. The unknowns in these equations, however, must only involve the internal moments acting at the ends of the columns, since the slope deflection equations involve these moments.

Example 4

- Determine the moments at each joint of the frame shown in the figure below. EI is constant.



Solution 4

Slope-Deflection Equations. Three spans must be considered in this problem: AB, BC, and CD. Since the spans are fixed supported at A and D, Eq. 11–8 applies for the solution.

From the table on the inside back cover, the FEMs for BC are

$$(\text{FEM})_{BC} = -\frac{5wL^2}{96} = -\frac{5(24)(8)^2}{96} = -80\ \text{kN}\cdot\text{m}$$

$$(\text{FEM})_{CB} = \frac{5wL^2}{96} = \frac{5(24)(8)^2}{96} = 80\ \text{kN}\cdot\text{m}$$

Note that $\theta_A = \theta_D = 0$ and $\psi_{AB} = \psi_{BC} = \psi_{CD} = 0$, since no sidesway will occur.

Applying Eq. 11–8, we have

$$M_N = 2Ek(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

$$M_{AB} = 2E\left(\frac{I}{12}\right)[2(0) + \theta_B - 3(0)] + 0$$

$$M_{AB} = 0.1667EI\theta_B \qquad (1)$$

$$M_{BA} = 2E\left(\frac{I}{12}\right)[2\theta_B + 0 - 3(0)] + 0$$

$$M_{BA} = 0.333EI\theta_B \qquad (2)$$

$$M_{BC} = 2E\left(\frac{I}{8}\right)[2\theta_B + \theta_C - 3(0)] - 80$$

$$M_{BC} = 0.5EI\theta_B + 0.25EI\theta_C - 80 \qquad (3)$$

$$M_{CB} = 2E\left(\frac{I}{8}\right)[2\theta_C + \theta_B - 3(0)] + 80$$

$$M_{CB} = 0.5EI\theta_C + 0.25EI\theta_B + 80 \qquad (4)$$

$$M_{CD} = 2E\left(\frac{I}{12}\right)[2\theta_C + 0 - 3(0)] + 0$$

$$M_{CD} = 0.333EI\theta_C \qquad (5)$$

$$M_{DC} = 2E\left(\frac{I}{12}\right)[2(0) + \theta_C - 3(0)] + 0$$

$$M_{DC} = 0.1667EI\theta_C \qquad (6)$$

Solution 4

Equilibrium Equations. The preceding six equations contain eight unknowns. The remaining two equilibrium equations come from moment equilibrium at joints B and C, Fig. 11–16b. We have

$$M_{BA} + M_{BC} = 0 \qquad (7)$$
$$M_{CB} + M_{CD} = 0 \qquad (8)$$

To solve these eight equations, substitute Eqs. (2) and (3) into Eq. (7) and substitute Eqs. (4) and (5) into Eq. (8). We get

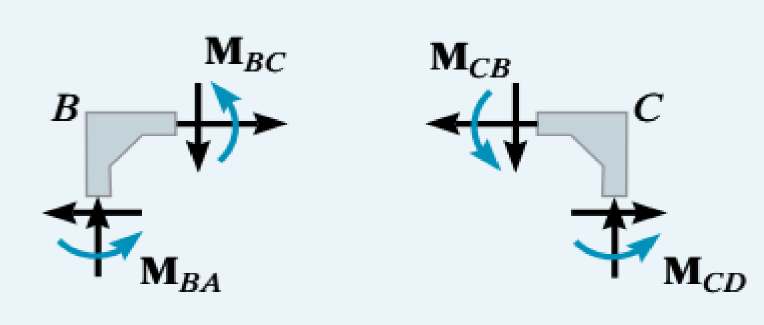
$$0.833EI\theta_B + 0.25EI\theta_C = 80$$
$$0.833EI\theta_C + 0.25EI\theta_B = -80$$

Solving simultaneously yields

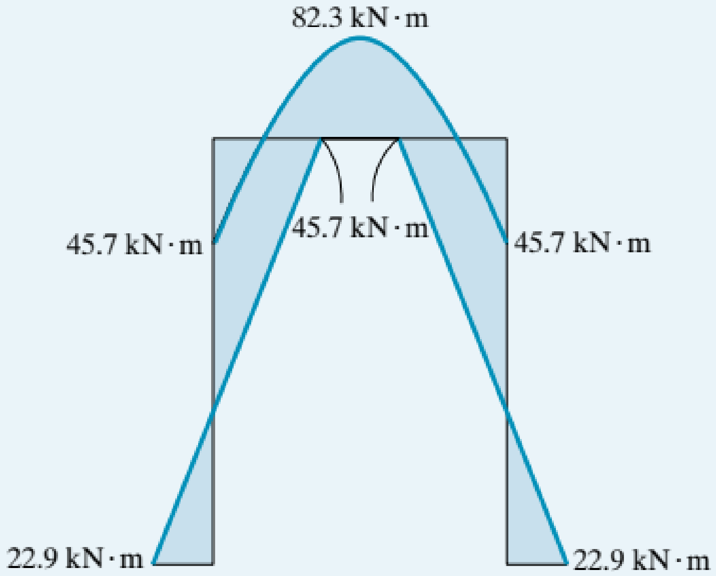
$$\theta_B = -\theta_C = \frac{137.1}{EI}$$

which conforms with the way the frame deflects as shown in Fig. 11–16a. Substituting into Eqs. (1)–(6), we get

$$M_{AB} = 22.9 \text{ kN} \cdot \text{m} \qquad \textit{Ans.}$$
$$M_{BA} = 45.7 \text{ kN} \cdot \text{m} \qquad \textit{Ans.}$$
$$M_{BC} = -45.7 \text{ kN} \cdot \text{m} \qquad \textit{Ans.}$$
$$M_{CB} = 45.7 \text{ kN} \cdot \text{m} \qquad \textit{Ans.}$$
$$M_{CD} = -45.7 \text{ kN} \cdot \text{m} \qquad \textit{Ans.}$$
$$M_{DC} = -22.9 \text{ kN} \cdot \text{m} \qquad \textit{Ans.}$$

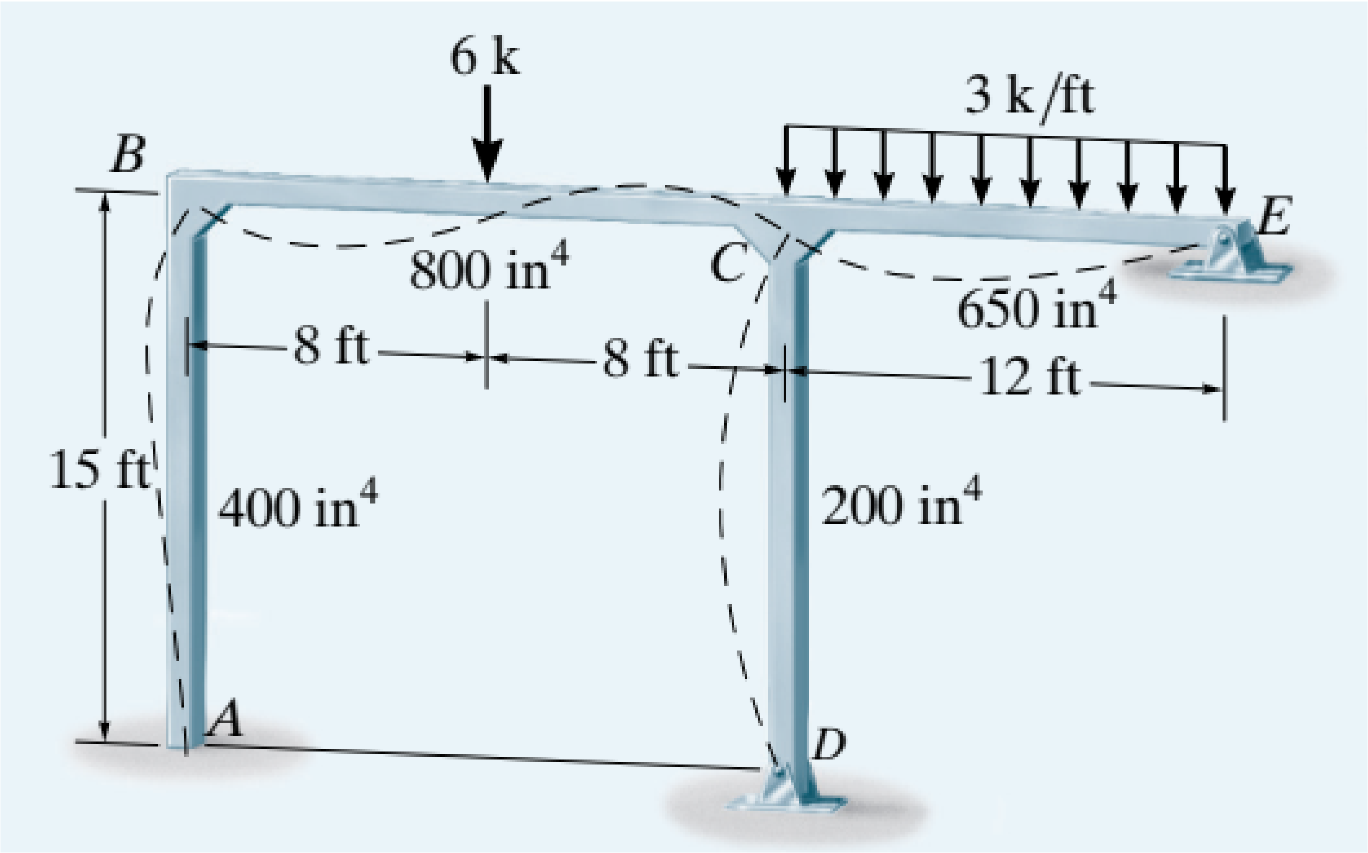


Using these results, the reactions at the ends of each member can be determined from the equations of equilibrium, and the moment diagram for the frame can be drawn, Fig. 11–16c.



Example 5

- Determine the internal moments at each joint of the frame shown in the figure below. The moment of inertia for each member is given in the figure. Take $E = 29(10^3)\ ksi$.



Solution 5

Slope-Deflection Equations. Four spans must be considered in this problem. Equation 11–8 applies to spans AB and BC, and Eq. 11–10 will be applied to CD and CE, because the ends at D and E are pinned.

Computing the member stiffnesses, we have

$$k_{AB} = \frac{400}{15(12)^4} = 0.001286 \text{ ft}^3 \qquad k_{CD} = \frac{200}{15(12)^4} = 0.000643 \text{ ft}^3$$

$$k_{BC} = \frac{800}{16(12)^4} = 0.002411 \text{ ft}^3 \qquad k_{CE} = \frac{650}{12(12)^4} = 0.002612 \text{ ft}^3$$

The FEMs due to the loadings are

$$(\text{FEM})_{BC} = -\frac{PL}{8} = -\frac{6(16)}{8} = -12 \text{ k} \cdot \text{ft}$$

$$(\text{FEM})_{CB} = \frac{PL}{8} = \frac{6(16)}{8} = 12 \text{ k} \cdot \text{ft}$$

$$(\text{FEM})_{CE} = -\frac{wL^2}{8} = -\frac{3(12)^2}{8} = -54 \text{ k} \cdot \text{ft}$$

Applying Eqs. 11–8 and 11–10 to the frame and noting that $\theta_A = 0$, $\psi_{AB} = \psi_{BC} = \psi_{CD} = \psi_{CE} = 0$ since no sidesway occurs, we have

$$M_N = 2Ek(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

$$M_{AB} = 2\left[29(10^3)(12)^2\right](0.001286)[2(0) + \theta_B - 3(0)] + 0$$

$$M_{AB} = 10740.7\theta_B \qquad (1)$$

$$M_{BA} = 2\left[29(10^3)(12)^2\right](0.001286)[2\theta_B + 0 - 3(0)] + 0$$

$$M_{BA} = 21\,481.5\theta_B \qquad (2)$$

$$M_{BC} = 2\left[29(10^3)(12)^2\right](0.002411)[2\theta_B + \theta_C - 3(0)] - 12$$

$$M_{BC} = 40\,277.8\theta_B + 20\,138.9\theta_C - 12 \qquad (3)$$

$$M_{CB} = 2\left[29(10^3)(12)^2\right](0.002411)[2\theta_C + \theta_B - 3(0)] + 12$$

$$M_{CB} = 20\,138.9\theta_B + 40\,277.8\theta_C + 12 \qquad (4)$$

$$M_N = 3Ek(\theta_N - \psi) + (\text{FEM})_N$$

$$M_{CD} = 3\left[29(10^3)(12)^2\right](0.000643)[\theta_C - 0] + 0 \qquad (5)$$

$$M_{CD} = 8055.6\theta_C$$

$$M_{CE} = 3\left[29(10^3)(12)^2\right](0.002612)[\theta_C - 0] - 54$$

$$M_{CE} = 32\,725.7\theta_C - 54 \qquad (6)$$

Solution 5

Equations of Equilibrium. These six equations contain eight unknowns. Two moment equilibrium equations can be written for joints B and C, Fig. 11–17b. We have

$$M_{BA} + M_{BC} = 0 \qquad (7)$$

$$M_{CB} + M_{CD} + M_{CE} = 0 \qquad (8)$$

In order to solve, substitute Eqs. (2) and (3) into Eq. (7), and Eqs. (4)–(6) into Eq. (8). This gives

$$61\,759.3\theta_B + 20\,138.9\theta_C = 12$$

$$20\,138.9\theta_B + 81\,059.0\theta_C = 42$$

Solving these equations simultaneously yields

$$\theta_B = 2.758(10^{-5})\text{ rad} \qquad \theta_C = 5.113(10^{-4})\text{ rad}$$

These values, being clockwise, tend to distort the frame as shown in Fig. 11–17a. Substituting these values into Eqs. (1)–(6) and solving, we get

$$M_{AB} = 0.296\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

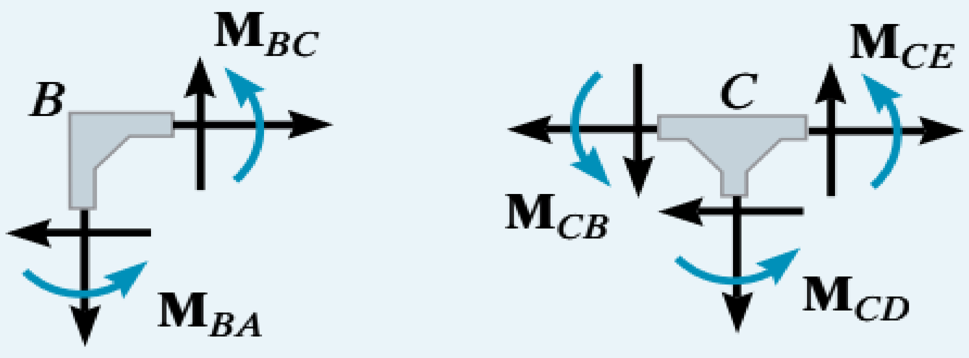
$$M_{BA} = 0.592\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{BC} = -0.592\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{CB} = 33.1\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

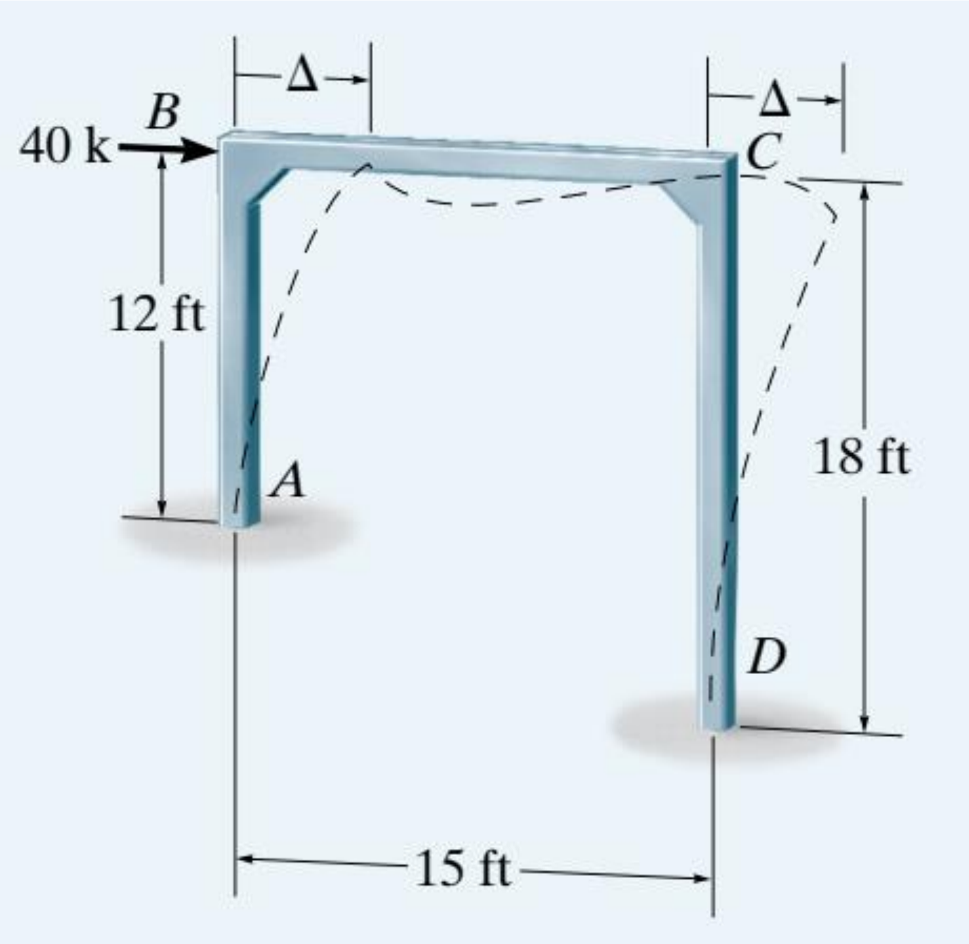
$$M_{CD} = 4.12\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{CE} = -37.3\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$



Example 6

- Determine the moments at each joint of the frame shown in the figure below. EI is constant.



Solution 6

Determine the moments at each joint of the frame shown in Fig. 11–19*a*. *EI* is constant.

SOLUTION

Slope-Deflection Equations. Since the ends *A* and *D* are fixed, Eq. 11–8 applies for all three spans of the frame. Sidesway occurs here since both the applied loading and the geometry of the frame are nonsymmetric. Here the load is applied directly to joint *B* and therefore no FEMs act at the joints. As shown in Fig. 11–19*a*, both joints *B* and *C* are assumed to be displaced an *equal amount* Δ. Consequently, $\psi_{AB} = \Delta/12$ and $\psi_{DC} = \Delta/18$. Both terms are positive since the cords of members *AB* and *CD* "rotate" clockwise. Relating ψ_{AB} to ψ_{DC}, we have $\psi_{AB} = (18/12)\psi_{DC}$. Applying Eq. 11–8 to the frame, we have

$$M_{AB} = 2E\left(\frac{I}{12}\right)\left[2(0) + \theta_B - 3\left(\frac{18}{12}\psi_{DC}\right)\right] + 0 = EI(0.1667\theta_B - 0.75\psi_{DC}) \qquad (1)$$

$$M_{BA} = 2E\left(\frac{I}{12}\right)\left[2\theta_B + 0 - 3\left(\frac{18}{12}\psi_{DC}\right)\right] + 0 = EI(0.333\theta_B - 0.75\psi_{DC}) \qquad (2)$$

$$M_{BC} = 2E\left(\frac{I}{15}\right)\left[2\theta_B + \theta_C - 3(0)\right] + 0 = EI(0.267\theta_B + 0.133\theta_C) \qquad (3)$$

Solution 6

$$M_{CB} = 2E\left(\frac{I}{15}\right)[2\theta_C + \theta_B - 3(0)] + 0 = EI(0.267\theta_C + 0.133\theta_B) \quad (4)$$

$$M_{CD} = 2E\left(\frac{I}{18}\right)[2\theta_C + 0 - 3\psi_{DC}] + 0 = EI(0.222\theta_C - 0.333\psi_{DC}) \quad (5)$$

$$M_{DC} = 2E\left(\frac{I}{18}\right)[2(0) + \theta_C - 3\psi_{DC}] + 0 = EI(0.111\theta_C - 0.333\psi_{DC}) \quad (6)$$

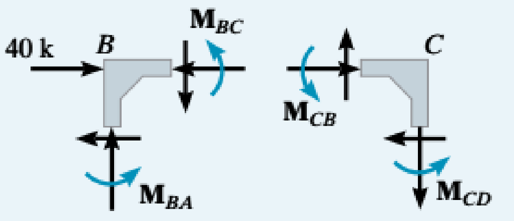
Equations of Equilibrium. The six equations contain nine unknowns. Two moment equilibrium equations for joints B and C, Fig. 11–19*b*, can be written, namely,

$$M_{BA} + M_{BC} = 0 \quad (7)$$

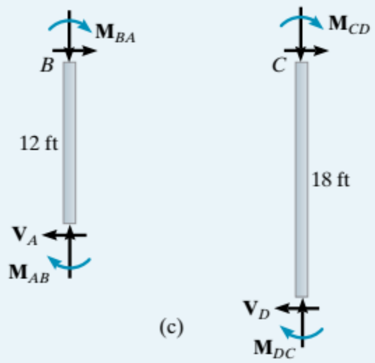
$$M_{CB} + M_{CD} = 0 \quad (8)$$

Since a horizontal displacement Δ occurs, we will consider summing forces on the *entire frame* in the x direction. This yields

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad 40 - V_A - V_D = 0$$



(b)



The horizontal reactions or column shears V_A and V_D can be related to the internal moments by considering the free-body diagram of each column separately, Fig. 11–19*c*. We have

$$\Sigma M_B = 0; \qquad V_A = -\frac{M_{AB} + M_{BA}}{12}$$

$$\Sigma M_C = 0; \qquad V_D = -\frac{M_{DC} + M_{CD}}{18}$$

Thus,

$$40 + \frac{M_{AB} + M_{BA}}{12} + \frac{M_{DC} + M_{CD}}{18} = 0 \quad (9)$$

In order to solve, substitute Eqs. (2) and (3) into Eq. (7), Eqs. (4) and (5) into Eq. (8), and Eqs. (1), (2), (5), (6) into Eq. (9). This yields

$$0.6\theta_B + 0.133\theta_C - 0.75\psi_{DC} = 0$$

$$0.133\theta_B + 0.489\theta_C - 0.333\psi_{DC} = 0$$

$$0.5\theta_B + 0.222\theta_C - 1.944\psi_{DC} = -\frac{480}{EI}$$

Solving simultaneously, we have

$$EI\theta_B = 438.81 \qquad EI\theta_C = 136.18 \qquad EI\psi_{DC} = 375.26$$

Finally, using these results and solving Eqs. (1)–(6) yields

$M_{AB} = -208\ \text{k}\cdot\text{ft}$ *Ans.*

$M_{BA} = -135\ \text{k}\cdot\text{ft}$ *Ans.*

$M_{BC} = 135\ \text{k}\cdot\text{ft}$ *Ans.*

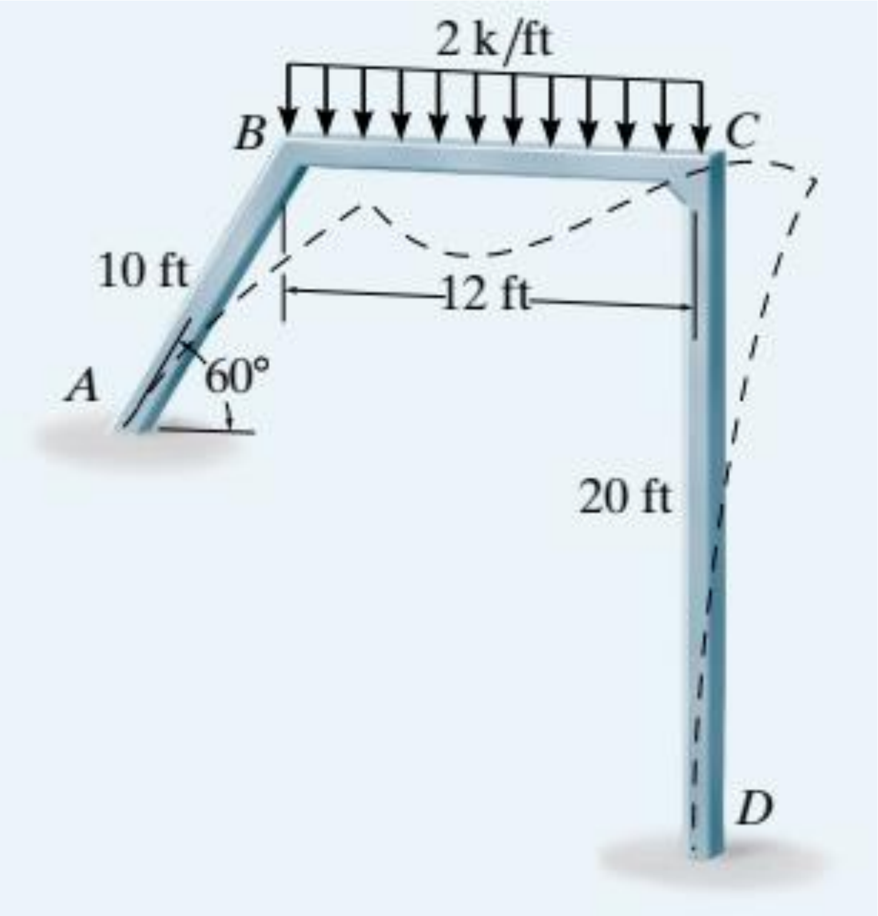
$M_{CB} = 94.8\ \text{k}\cdot\text{ft}$ *Ans.*

$M_{CD} = -94.8\ \text{k}\cdot\text{ft}$ *Ans.*

$M_{DC} = -110\ \text{k}\cdot\text{ft}$ *Ans.*

Example 7

- Determine the moments at each joint of the frame shown in the figure below. EI is constant for each member.



Solution 7

Slope-Deflection Equations. Equation 11–8 applies to each of the three spans. The FEMs are

$$(\text{FEM})_{BC} = -\frac{wL^2}{12} = -\frac{2(12)^2}{12} = -24\ \text{k}\cdot\text{ft}$$

$$(\text{FEM})_{CB} = \frac{wL^2}{12} = \frac{2(12)^2}{12} = 24\ \text{k}\cdot\text{ft}$$

The sloping member AB causes the frame to sidesway to the right as shown in Fig. 11–22*a*. As a result, joints B and C are subjected to both rotational *and* linear displacements. The linear displacements are shown in Fig. 11–22*b*, where B moves Δ_1 to B' and C moves Δ_3 to C'. These displacements cause the members' cords to rotate ψ_1, ψ_3 (clockwise) and $-\psi_2$ (counterclockwise) as shown.* Hence,

$$\psi_1 = \frac{\Delta_1}{10} \qquad \psi_2 = -\frac{\Delta_2}{12} \qquad \psi_3 = \frac{\Delta_3}{20}$$

As shown in Fig. 11–22*c*, the three displacements can be related. For example, $\Delta_2 = 0.5\Delta_1$ and $\Delta_3 = 0.866\Delta_1$. Thus, from the above equations we have

$$\psi_2 = -0.417\psi_1 \qquad \psi_3 = 0.433\psi_1$$

Using these results, the slope-deflection equations for the frame are

$$M_{AB} = 2E\left(\frac{I}{10}\right)[2(0) + \theta_B - 3\psi_1] + 0 \qquad (1)$$

$$M_{BA} = 2E\left(\frac{I}{10}\right)[2\theta_B + 0 - 3\psi_1] + 0 \qquad (2)$$

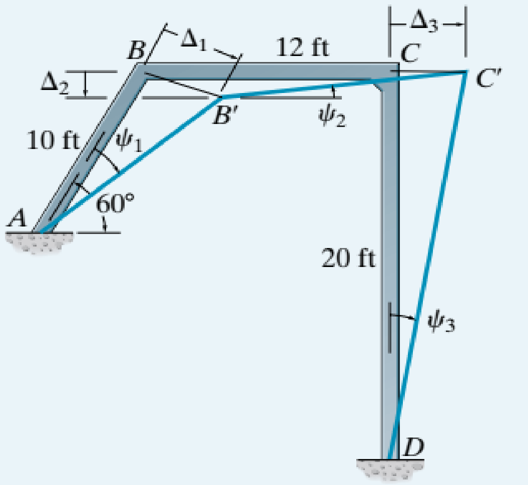
$$M_{BC} = 2E\left(\frac{I}{12}\right)[2\theta_B + \theta_C - 3(-0.417\psi_1)] - 24 \qquad (3)$$

$$M_{CB} = 2E\left(\frac{I}{12}\right)[2\theta_C + \theta_B - 3(-0.417\psi_1)] + 24 \qquad (4)$$

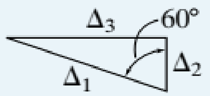
$$M_{CD} = 2E\left(\frac{I}{20}\right)[2\theta_C + 0 - 3(0.433\psi_1)] + 0 \qquad (5)$$

$$M_{DC} = 2E\left(\frac{I}{20}\right)[2(0) + \theta_C - 3(0.433\psi_1)] + 0 \qquad (6)$$

These six equations contain nine unknowns.



(b)



(c)

Solution 7

Equations of Equilibrium. Moment equilibrium at joints B and C yields

$$M_{BA} + M_{BC} = 0 \tag{7}$$

$$M_{CD} + M_{CB} = 0 \tag{8}$$

The necessary third equilibrium equation can be obtained by summing moments about point O on the entire frame, Fig. 11–22*d*. This eliminates the unknown normal forces $\mathbf{N}_A$ and $\mathbf{N}_D$, and therefore

$\circlearrowleft + \Sigma M_O = 0;$

$$M_{AB} + M_{DC} - \left(\frac{M_{AB} + M_{BA}}{10}\right)(34) - \left(\frac{M_{DC} + M_{CD}}{20}\right)(40.78) - 24(6) = 0$$

$$-2.4M_{AB} - 3.4M_{BA} - 2.04M_{CD} - 1.04M_{DC} - 144 = 0 \tag{9}$$

Substituting Eqs. (2) and (3) into Eq. (7), Eqs. (4) and (5) into Eq. (8), and Eqs. (1), (2), (5), and (6) into Eq. (9) yields

$$0.733\theta_B + 0.167\theta_C - 0.392\psi_1 = \frac{24}{EI}$$

$$0.167\theta_B + 0.533\theta_C + 0.0784\psi_1 = -\frac{24}{EI}$$

$$-1.840\theta_B - 0.512\theta_C + 3.880\psi_1 = \frac{144}{EI}$$

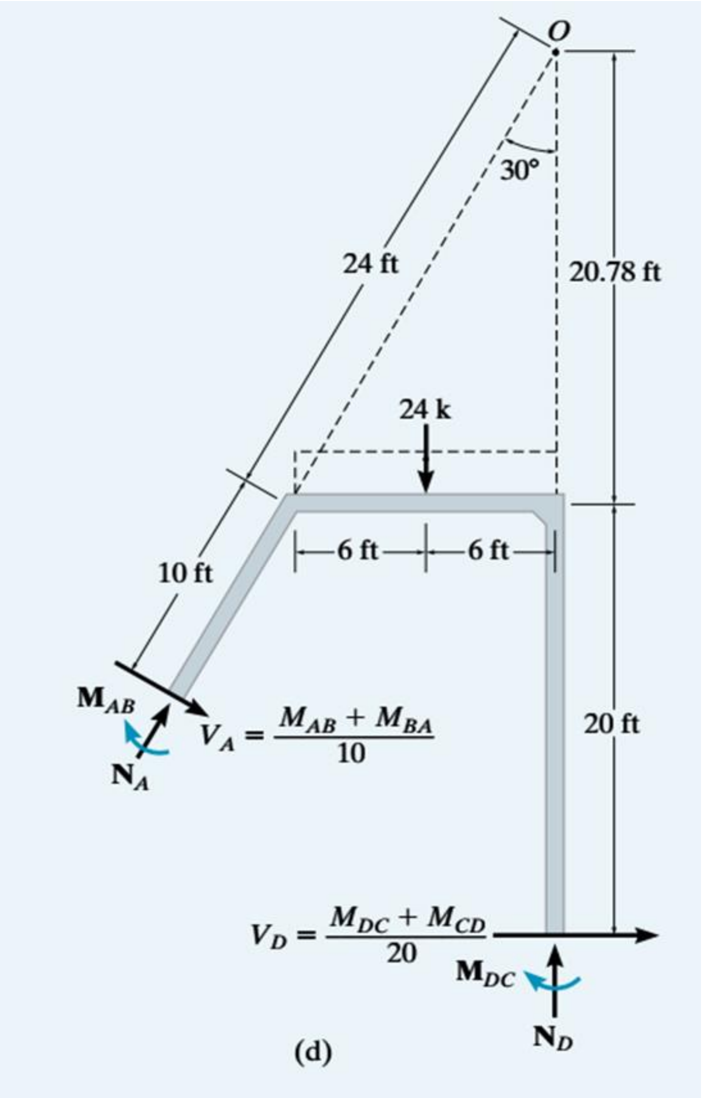
Solving these equations simultaneously yields

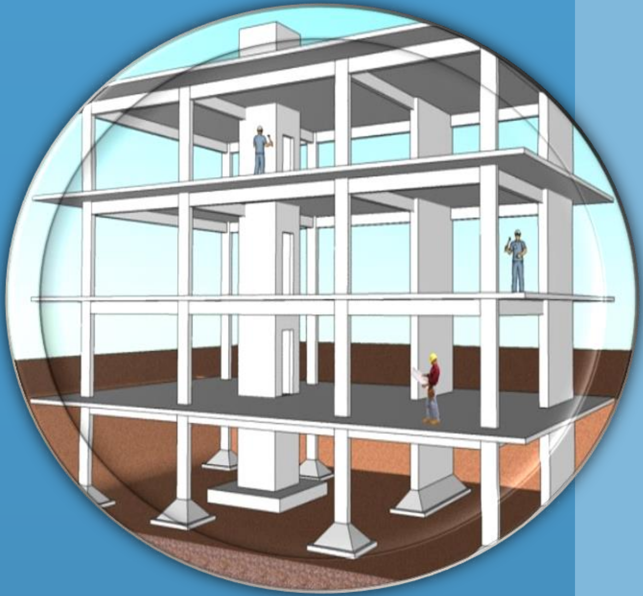
$$EI\theta_B = 87.67 \qquad EI\theta_C = -82.3 \qquad EI\psi_1 = 67.83$$

Substituting these values into Eqs. (1)–(6), we have

$M_{AB} = -23.2\ \text{k}\cdot\text{ft}$ $\quad M_{BC} = 5.63\ \text{k}\cdot\text{ft}$ $\quad M_{CD} = -25.3\ \text{k}\cdot\text{ft}$ *Ans.*

$M_{BA} = -5.63\ \text{k}\cdot\text{ft}$ $\quad M_{CB} = 25.3\ \text{k}\cdot\text{ft}$ $\quad M_{DC} = -17.0\ \text{k}\cdot\text{ft}$ *Ans.*





Thank You!