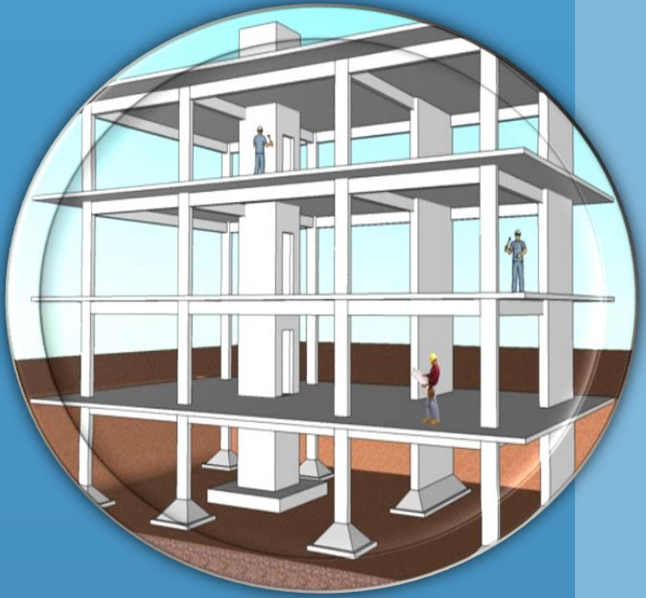
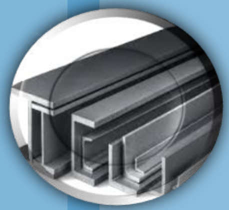
UNIVERSITY OF ZAMBIA
School of Engineering
Department of Civil & Environmental Eng.





CEE 3222: THEORY OF STRUCTURES

Lecture 4.3

DISPLACEMENT METHODS FOR SIS – MOMENT DISTRIBUTION METHOD

Mr. MWABA MSc, B.Eng., R.Eng., PEIZ,

Contents

Introduction

- For a coplanar structure there are at most three equilibrium equations for each part, so that if there is a total of n parts and r force and moment reaction components, we have

$$r = 3n, \text{ statically determinate}$$
$$r > 3n, \text{ statically indeterminate}$$

- This indeterminacy may arise as a result of:-
  - Added supports
  - Added members
  - General form of the structure (eg RC)

Moment Distribution Method

- The method of analyzing beams and frames using moment distribution was developed by Hardy Cross, in 1930.
- It is a method of successive approximations that may be carried out to any desired degree of accuracy.
- The method begins by assuming each joint of a structure is fixed. Then, by unlocking and locking each joint in succession, the internal moments at the joints are "distributed" and balanced until the joints have rotated to their final or nearly final positions.

fundamental process of moment distribution follows the same procedure as any displacement method. There the process is to establish load-displacement relations at each joint and then satisfy joint equilibrium requirements by determining the correct angular displacement for the joint (compatibility). Here, however, the equilibrium and compatibility of rotation at the joint is satisfied *directly*, using a "moment balance" process that incorporates the load-deflection relations (stiffness factors).

Moment Distribution Method

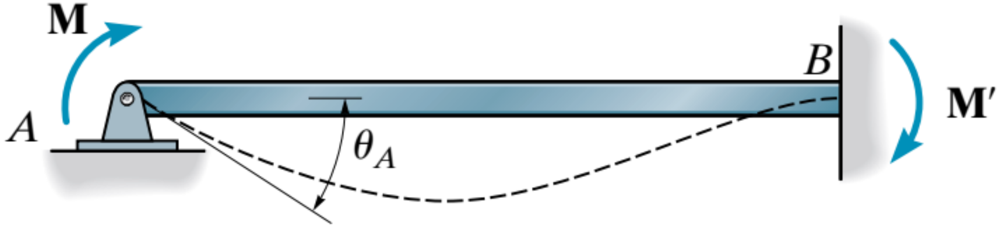
- **Sign Convention.**
  Clockwise moments are positive, whereas counterclockwise moments are negative

- **Fixed-End Moments (FEMs).**
  The moments at the "walls" or fixed joints of a loaded member are called fixed-end moments. Check the next two slides for details

- **Member Stiffness Factor**
  For the beam below, we saw that , $M = (4EI/L)\,\theta_A$

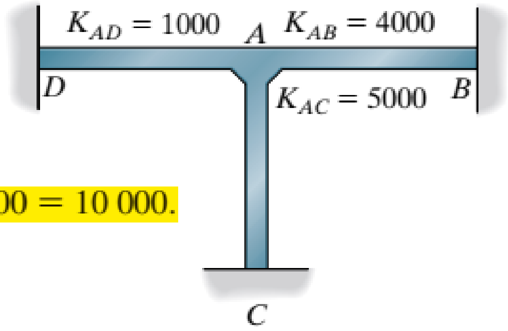
$$K = \frac{4EI}{L}$$

Far End Fixed



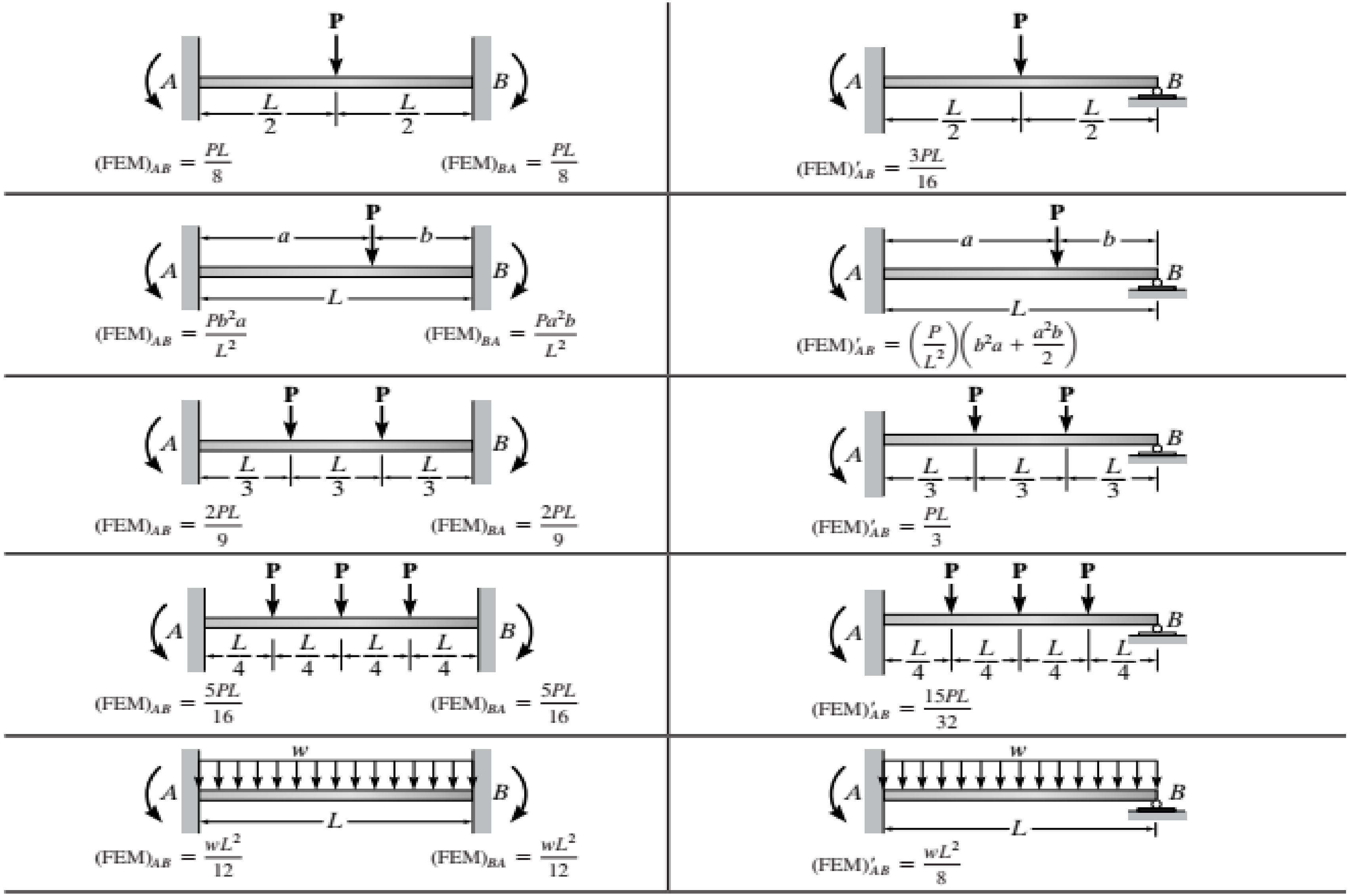
- **Joint Stiffness Factor.**
  If several members are fixed connected to a joint and each of their far ends is fixed, then by the principle of superposition, the total stiffness factor at the joint is the sum of the member stiffness factors at the joint,

$$K_T = \Sigma K$$

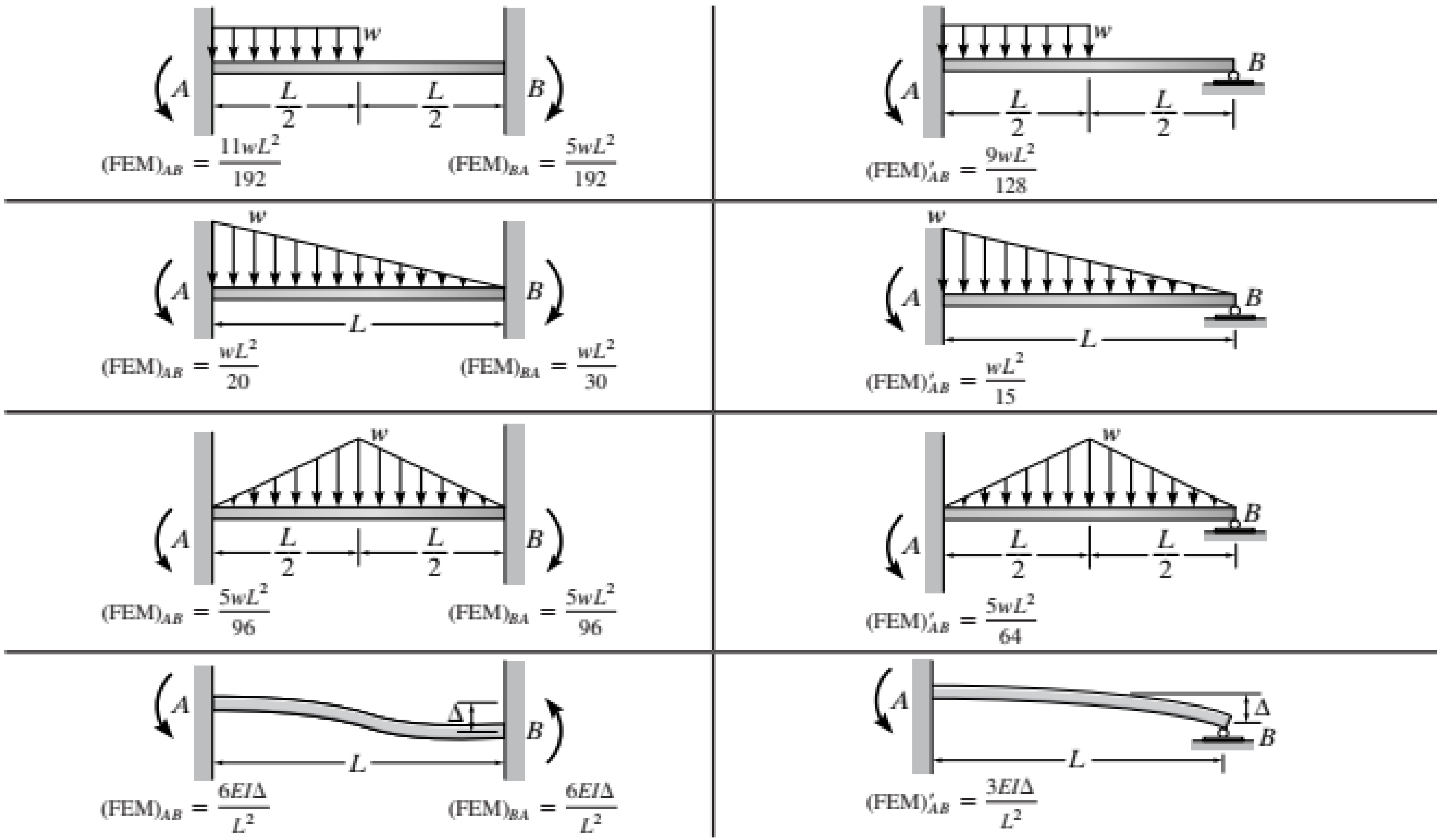


$K_T = \Sigma K = 4000 + 5000 + 1000 = 10\,000.$

Fixed end moments



Fixed end moments



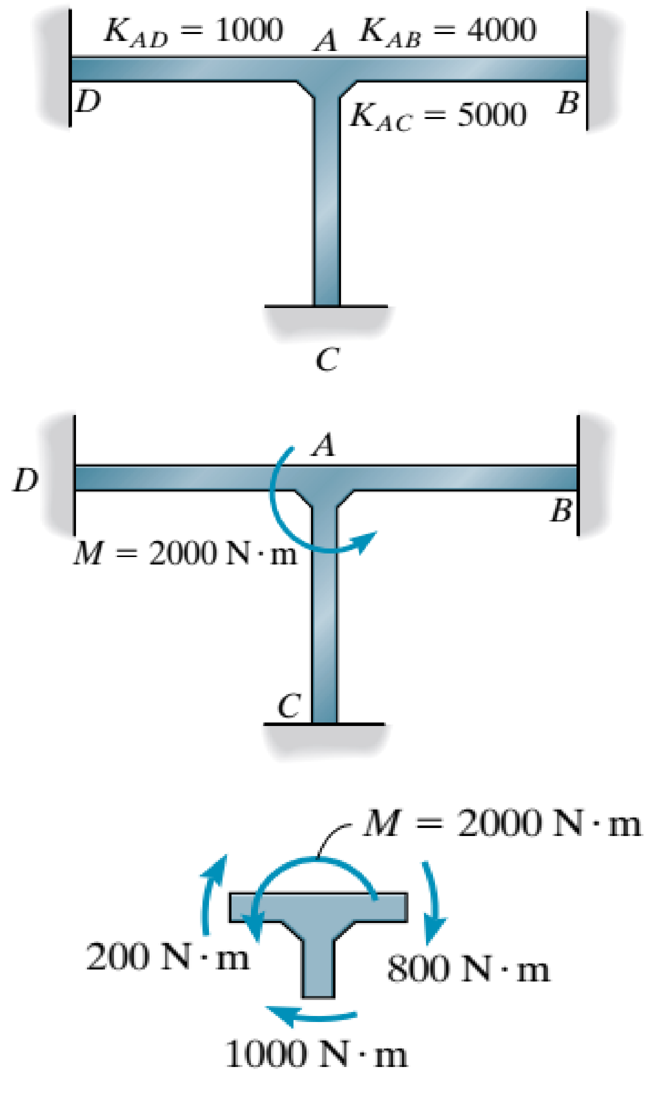
Moment Distribution Method- Distribution Factor

- If a moment M is applied to a fixed connected joint, the connecting members will each supply a portion of the resisting moment necessary to satisfy moment equilibrium at the joint.
- That fraction of the total resisting moment supplied by the member is called the distribution factor (DF).

$$\mathrm{DF} = \frac{K}{\Sigma K}$$

- For example, the DFs and moments for members AB, AC, and AD at joint A in the figure are:-

$$\mathrm{DF}_{AB} = 4000/10\,000 = 0.4 \qquad M_{AB} = 0.4(2000) = 800\ \mathrm{N\cdot m}$$
$$\mathrm{DF}_{AC} = 5000/10\,000 = 0.5 \qquad M_{AC} = 0.5(2000) = 1000\ \mathrm{N\cdot m}$$
$$\mathrm{DF}_{AD} = 1000/10\,000 = 0.1 \qquad M_{AD} = 0.1(2000) = 200\ \mathrm{N\cdot m}$$



(c)

Moment Distribution Method

- Member Relative-Stiffness Factor.

If the members of a frame or beam are made from the same material so its modulus of elasticity E will be the same for all the members. Then the common factor 4E will cancel out and henve we remain with the relative stiffness factor K_R

$$K = \frac{4EI}{L}$$

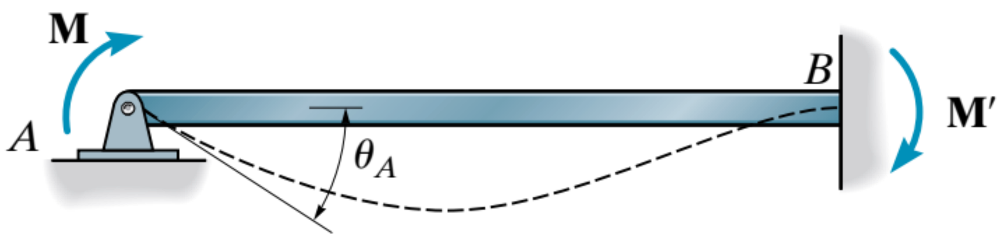
Far End Fixed

$$K_R = \frac{I}{L}$$

Far End Fixed

- Carry-Over Factor

Consider the beam below



$$M_{AB} = (4EI/L)\,\theta_A \qquad M_{BA} = (2EI/L)\,\theta_A$$

$$M_{BA} = M_{AB}/2$$

$$\mathbf{M}' = \tfrac{1}{2}\mathbf{M}$$

- The carry-over factor represents the fraction of M that is “carried over” from the pin to the wall.
- Hence, in the case of a beam with the far end fixed, the carry-over factor is $+\frac{1}{2}$. The plus sign indicates both moments act in the same direction.

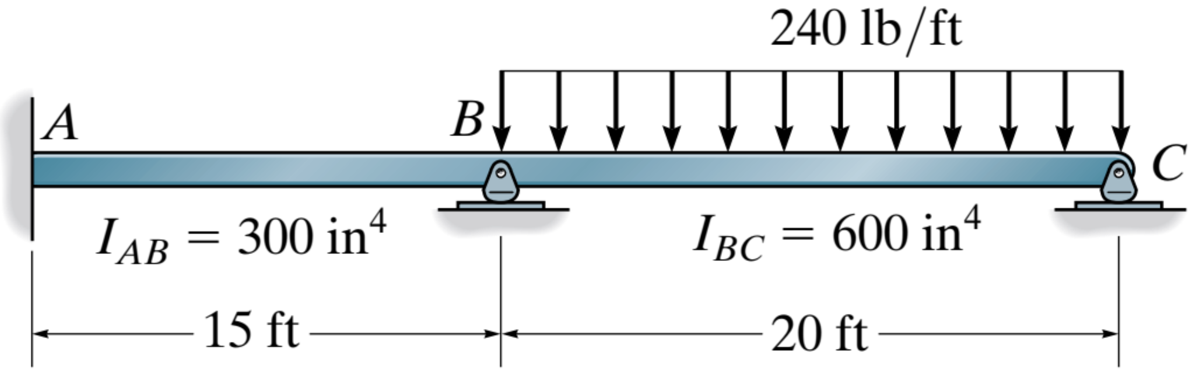
Moment Distribution for Beams

- Moment distribution is based on the principle of successively locking and unlocking the joints of a structure in order to allow the moments at the joints to be distributed and balanced.

- Consider this beam with a constant E. First we determine the stiffness factors K. Since in theory it would take an "infinite" size moment to rotate the wall one radian, the wall stiffness factor is infinite ∞. So we determine only on either side of Joint B

$$K_{BA} = \frac{4E(300)}{15} = 4E(20)\ \text{in}^4/\text{ft}$$

$$K_{BC} = \frac{4E(600)}{20} = 4E(30)\ \text{in}^4/\text{ft}$$



Moment Distribution for Beams- Joint B

- Next we determine the Distribution factors DFs (unitless)

$$\text{DF}_{BA} = \frac{4E(20)}{4E(20) + 4E(30)} = 0.4$$

$$\text{DF}_{BC} = \frac{4E(30)}{4E(20) + 4E(30)} = 0.6$$

- For the fixed wall joints A and C, the DFs are 0 as proved below

$$\text{DF}_{AB} = \frac{4E(20)}{\infty + 4E(20)} = 0$$

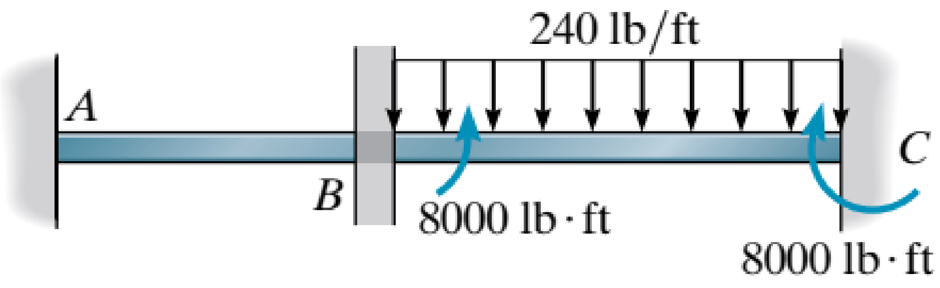
$$\text{DF}_{CB} = \frac{4E(30)}{\infty + 4E(30)} = 0$$

- Next we determine the Fixed end Moments

$$(\text{FEM})_{BC} = -\frac{wL^2}{12} = -\frac{240(20)^2}{12} = -8000 \text{ lb} \cdot \text{ft}$$

$$(\text{FEM})_{CB} = \frac{wL^2}{12} = \frac{240(20)^2}{12} = 8000 \text{ lb} \cdot \text{ft}$$

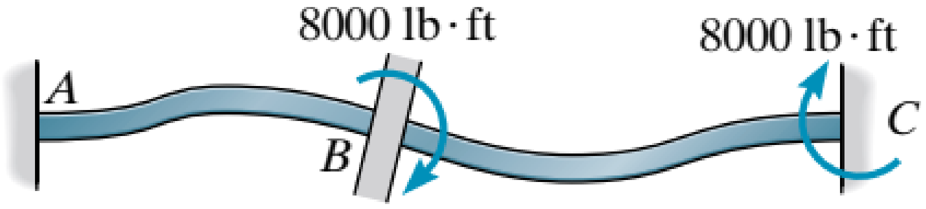
Moment Distribution for Beams - Joint B



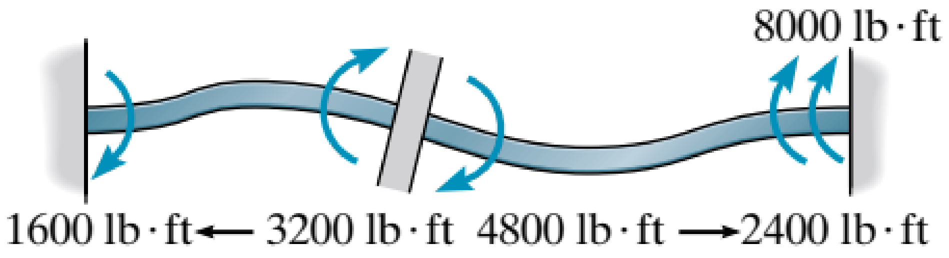
joint *B* held fixed

- First assume that B is fixed, find the fixed end moment (FEM) at B = -8000.

- To make joint B in Equilibrium, apply an equal but opposite moment to B. But Distribute it according to the Distribution factors (DFs)



correction moment applied to joint *B*



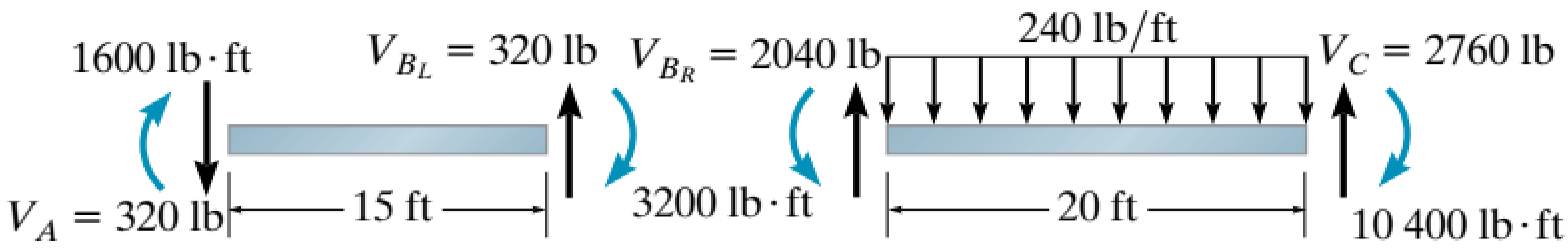
moment at *B* distributed

- Finally, due to the released rotation that takes place at B, these moments must be "carried over (CO)" since moments are developed at the far ends of the span.

Moment Distribution for Beams - Joint B

Joint	A	B		C
Member	AB	BA	BC	CB
DF	0	0.4	0.6	0
FEM Dist,CO	 1600 ←	 3200	−8000 4800 →	8000 2400
ΣM	1600	3200	−3200	10 400

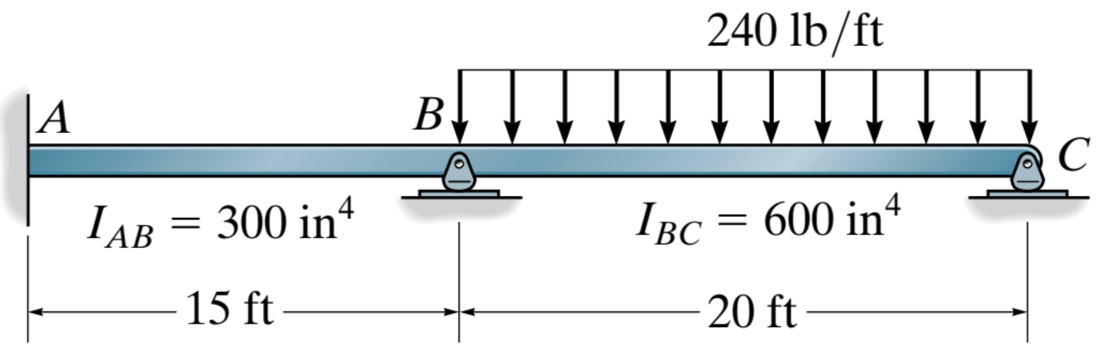
- With the end moments known, the end shears have been computed from the equations of equilibrium applied to each of these spans.



Moment Distribution for Beams- Joint C

- Consider now the same beam, except the support at C is a rocker as in the figure. In this case only one member is at joint C, so the distribution factor for member CB at joint C is

$$DF_{CB} = \frac{4E(30)}{4E(30)} = 1$$



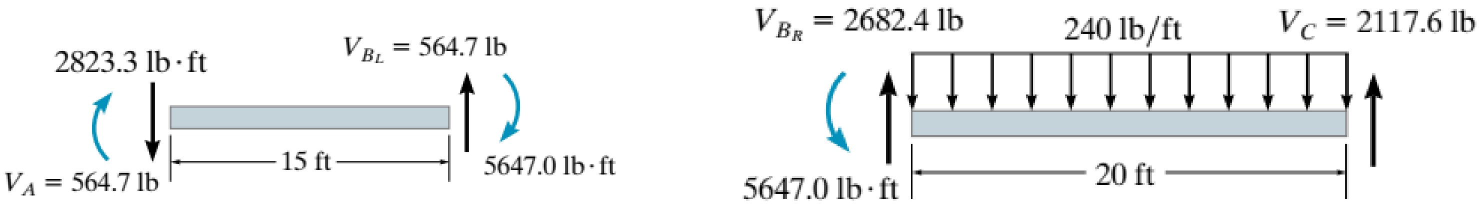
- The other DFs and FEMs are as computed before. (line 1 and 2).
- We have locked both joint B and C. Lets unlock joint C. It already has a +8000 moment, to bring it to equilibrium, we apply a -8000 moment and distribute it. Line 3 shows both the distributed (-8000) and carried over (-4000) moments.
- If we lock C and unlock B, we have total moment of -12000 at B, Thus we apply a correction moment of 12000 and distribute it as shown in Line 4. Then not that half of the moment is carried over to a fixed wall at A and the roller at C since joint B has rotated

Joint	*A*	*B*		*C*	
Member	*AB*	*BA*	*BC*	*CB*	
DF	0	0.4	0.6	1	1
FEM			−8000	8000	2
			−4000	−8000	3
	2400	4800	7200	3600	4
			−1800	−3600	5
	360	720	1080	540	6
			−270	−540	7
	54	108	162	81	8
			−40.5	−81	9
	8.1	16.2	24.3	12.2	10
			−6.1	−12.2	11
	1.2	2.4	3.6	1.8	12
			−0.9	−1.8	13
		0.4	0.5		14
ΣM	2823.3	5647.0	−5647.0	0	15

Moment Distribution for Beams

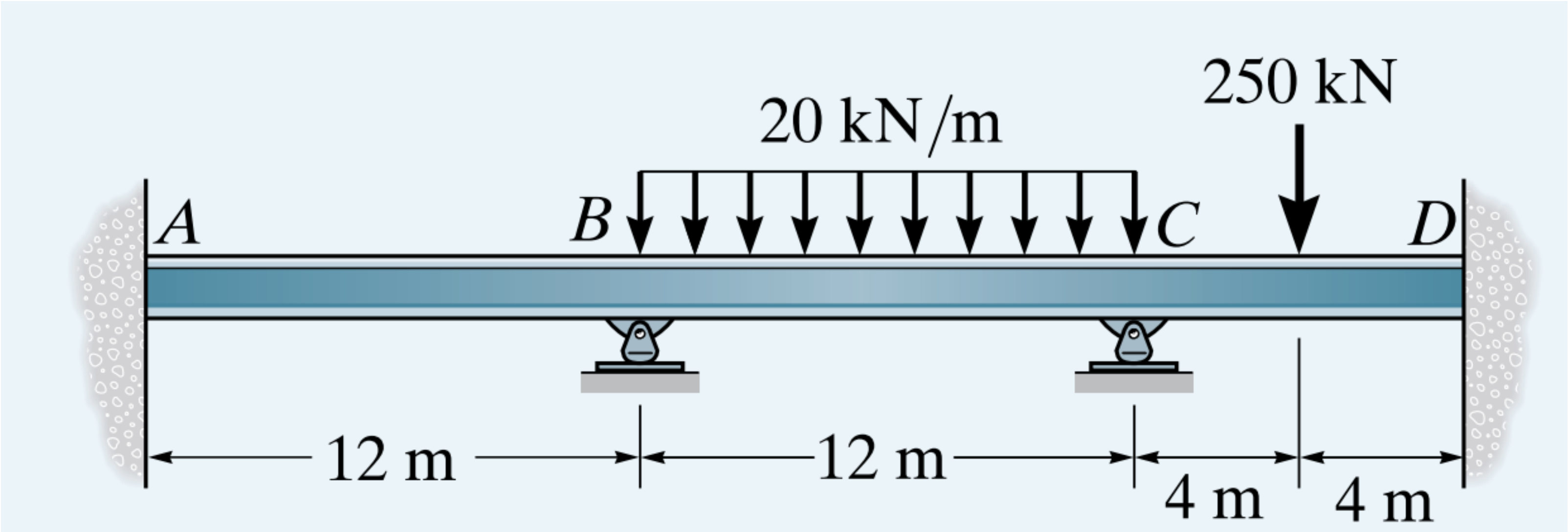
- It is also possible to apply it to all joints at the same time.
- Start by fixing all the joints and then balancing and distributing the fixed end moments at both joints B and C (line 3).
- Unlocking joint B and C simultaneously, the moments are carried to the ends of the span as shown in line 4
- Again the joints are relocked, and the moments are balanced and distributed, line 5.
- Unlocking the joints once again allows the moments to be carried over, as shown in line 6.
- This method is more efficient even though the convergence is slower. The rest is found from the FBD shown below

Joint	A	B		C	
Member	AB	BA	BC	CB	
DF	0	0.4	0.6	1	1
FEM			−8000	8000	2
Dist.		3200	4800	−8000	3
CO	1600		−4000	2400	4
Dist.		1600	2400	−2400	5
CO	800		−1200	1200	6
Dist.		480	720	−1200	7
CO	240		−600	360	8
Dist.		240	360	−360	9
CO	120		−180	180	10
Dist.		72	108	−180	11
CO	36		−90	54	12
Dist.		36	54	−54	13
CO	18		−27	27	14
Dist.		10.8	16.2	−27	15
CO	5.4		−13.5	8.1	16
Dist.		5.4	8.1	−8.1	17
CO	2.7		−4.05	4.05	18
Dist.		1.62	2.43	−4.05	19
CO	0.81		−2.02	1.22	20
Dist.		0.80	1.22	−1.22	21
CO	0.40		−0.61	0.61	22
Dist.		0.24	0.37	−0.61	23
ΣM	2823	5647	−5647	0	24



Example 1

- Determine the internal moments at each support of the beam shown in the figure below. *EI* is constant.



Solution 1

The distribution factors at each joint must be computed first.* The stiffness factors for the members are

$$K_{AB} = \frac{4EI}{12} \qquad K_{BC} = \frac{4EI}{12} \qquad K_{CD} = \frac{4EI}{8}$$

Therefore,

$$\mathrm{DF}_{AB} = \mathrm{DF}_{DC} = 0 \qquad \mathrm{DF}_{BA} = \mathrm{DF}_{BC} = \frac{4EI/12}{4EI/12 + 4EI/12} = 0.5$$

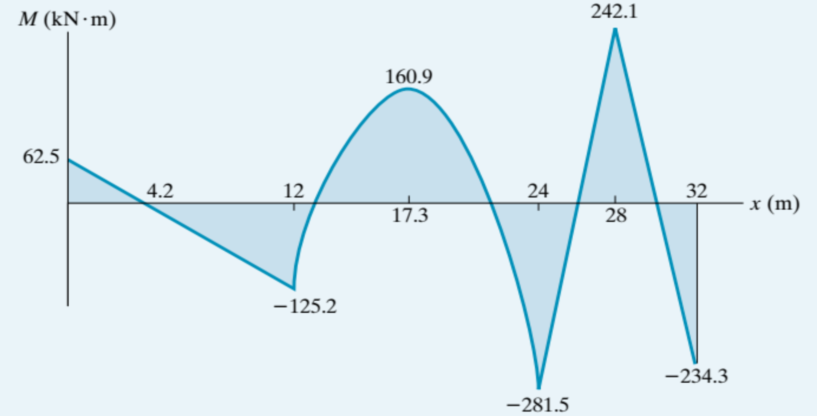
$$\mathrm{DF}_{CB} = \frac{4EI/12}{4EI/12 + 4EI/8} = 0.4 \quad \mathrm{DF}_{CD} = \frac{4EI/8}{4EI/12 + 4EI/8} = 0.6$$

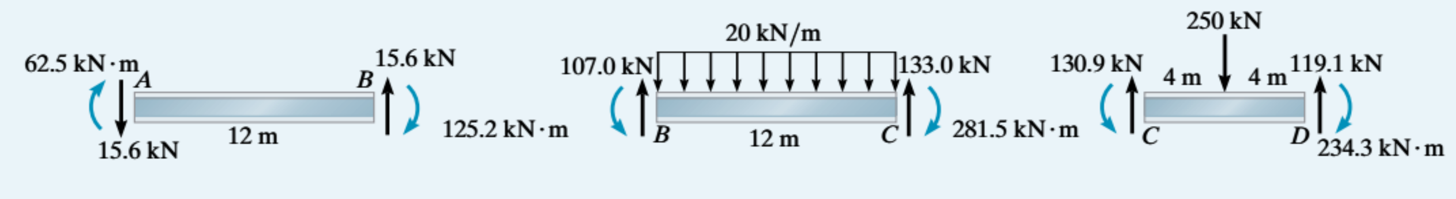
The fixed-end moments are

$$(\mathrm{FEM})_{BC} = -\frac{wL^2}{12} = \frac{-20(12)^2}{12} = -240\ \mathrm{kN\cdot m} \qquad (\mathrm{FEM})_{CB} = \frac{wL^2}{12} = \frac{20(12)^2}{12} = 240\ \mathrm{kN\cdot m}$$

$$(\mathrm{FEM})_{CD} = -\frac{PL}{8} = \frac{-250(8)}{8} = -250\ \mathrm{kN\cdot m} \qquad (\mathrm{FEM})_{DC} = \frac{PL}{8} = \frac{250(8)}{8} = 250\ \mathrm{kN\cdot m}$$

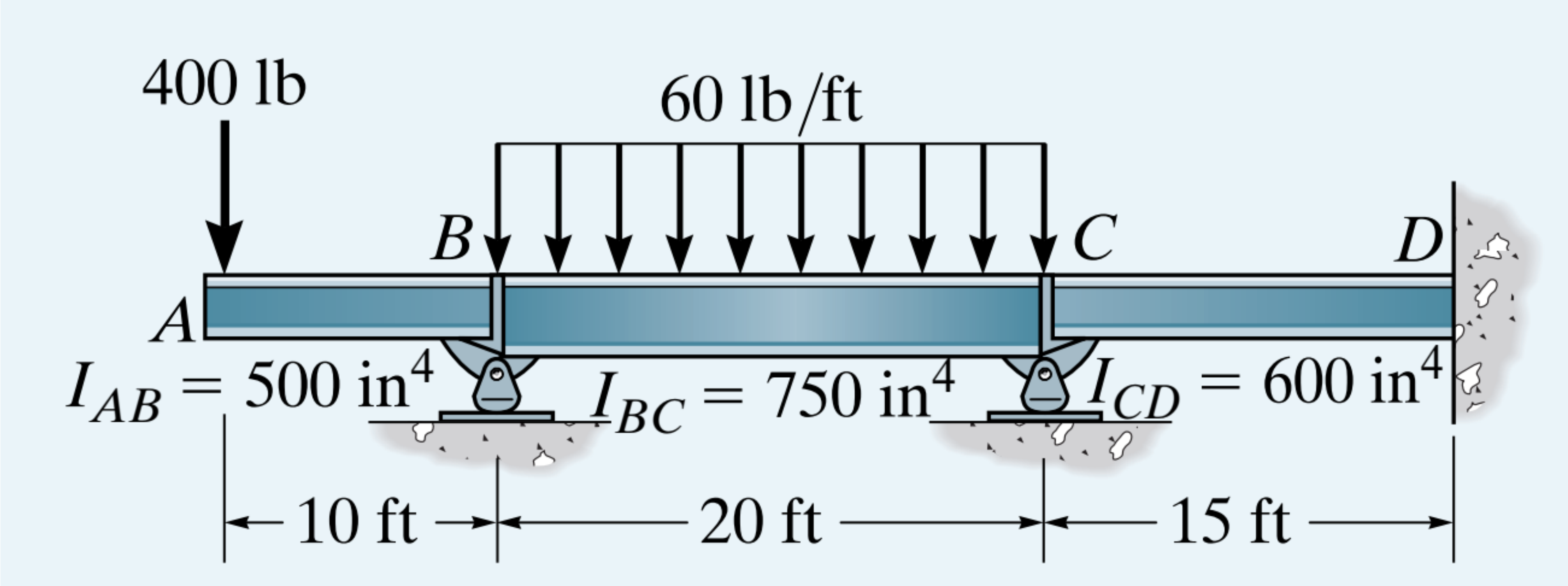
Joint	A	B		C		D	1
Member	AB	BA	BC	CB	CD	DC	2
DF	0	0.5	0.5	0.4	0.6	0	3
FEM			−240	240	−250	250	4
Dist.		120	120	4	6		5
CO	60		2	60		3	6
Dist.		−1	−1	−24	−36		7
CO	−0.5		−12	−0.5		−18	8
Dist.		6	6	0.2	0.3		9
CO	3		0.1	3		0.2	10
Dist.		−0.05	−0.05	−1.2	−1.8		11
CO	−0.02		−0.6	−0.02		−0.9	12
Dist.		0.3	0.3	0.01	0.01		13
ΣM	62.5	125.2	−125.2	281.5	−281.5	234.3	14





Example 2

- Determine the internal moment at each support of the beam shown in the figure. The moment of inertia of each span is indicated.



Solution 2

In this problem a moment does not get distributed in the overhanging span AB, and so the distribution factor $(\mathrm{DF})_{BA} = 0$. The stiffness of span BC is based on $4EI/L$ since the pin rocker is not at the far end of the beam. The stiffness factors, distribution factors, and fixed-end moments are computed as follows:

$$K_{BC} = \frac{4E(750)}{20} = 150E \qquad K_{CD} = \frac{4E(600)}{15} = 160E$$

$$\mathrm{DF}_{BC} = 1 - (\mathrm{DF})_{BA} = 1 - 0 = 1$$

$$\mathrm{DF}_{CB} = \frac{150E}{150E + 160E} = 0.484$$

$$\mathrm{DF}_{CD} = \frac{160E}{150E + 160E} = 0.516$$

$$\mathrm{DF}_{DC} = \frac{160E}{\infty + 160E} = 0$$

Due to the overhang,

$$(\mathrm{FEM})_{BA} = 400 \text{ lb}(10 \text{ ft}) = 4000 \text{ lb}\cdot\text{ft}$$

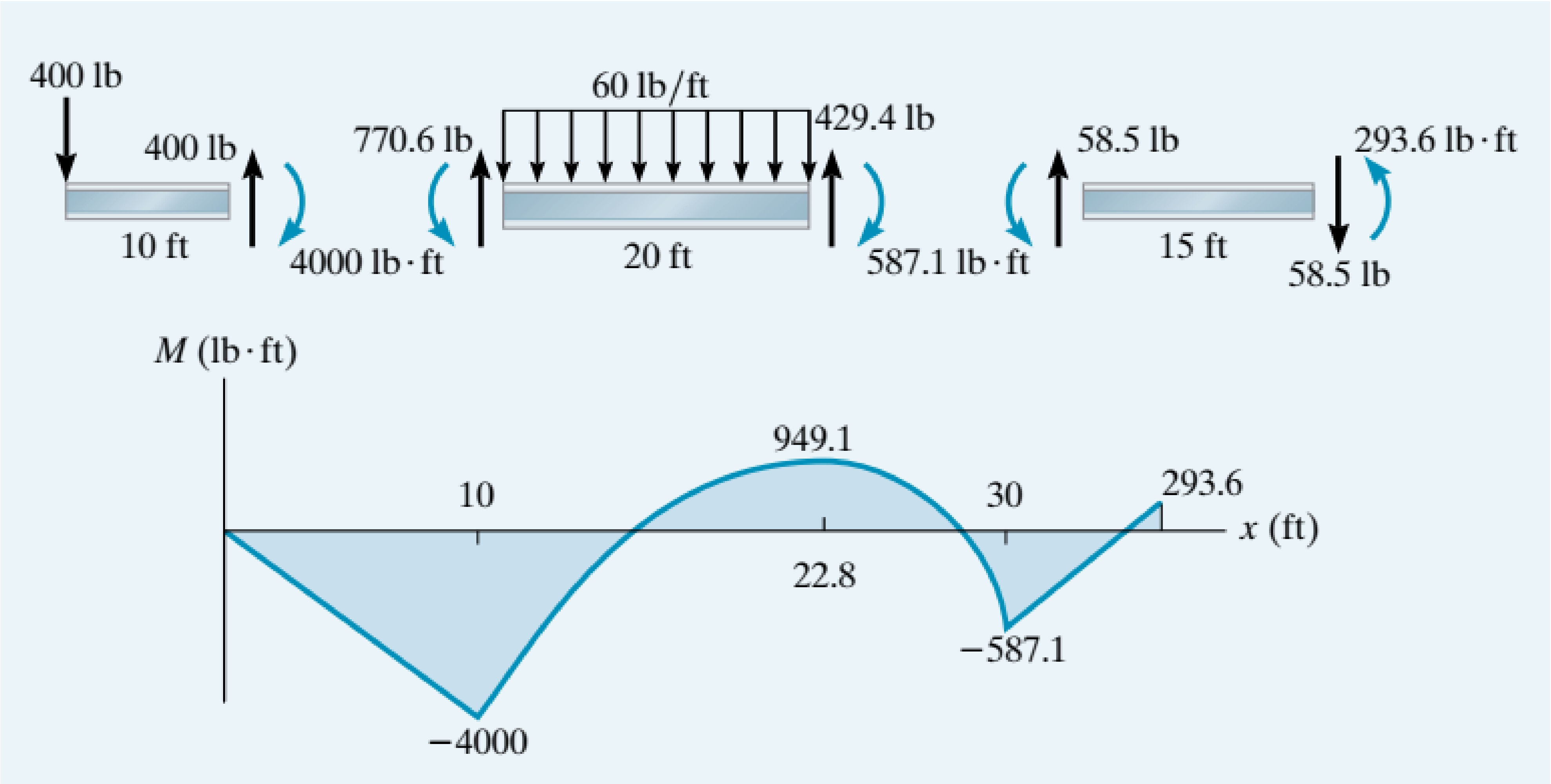
$$(\mathrm{FEM})_{BC} = -\frac{wL^2}{12} = -\frac{60(20)^2}{12} = -2000 \text{ lb}\cdot\text{ft}$$

$$(\mathrm{FEM})_{CB} = \frac{wL^2}{12} = \frac{60(20)^2}{12} = 2000 \text{ lb}\cdot\text{ft}$$

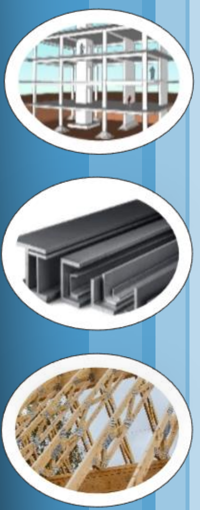
These values are listed on the fourth line of the table, Fig. 12–8*b*. The overhanging span requires the internal moment to the left of B to be $+4000$ lb · ft. Balancing at joint B requires an internal moment of -4000 lb · ft to the right of B. As shown on the fifth line of the table -2000 lb · ft is added to BC in order to satisfy this condition. The distribution and carry-over operations proceed in the usual manner as indicated.

Joint	B		C		D
Member		BC	CB	CD	DC
DF	0	1	0.484	0.516	0
FEM	4000	−2000	2000		
Dist.		−2000	−968	−1032	
CO		−484	−1000		−516
Dist.		484	484	516	
CO		242	242		258
Dist.		−242	−117.1	−124.9	
CO		−58.6	−121		−62.4
Dist.		58.6	58.6	62.4	
CO		29.3	29.3		31.2
Dist.		−29.3	−14.2	−15.1	
CO		−7.1	−14.6		−7.6
Dist.		7.1	7.1	7.6	
CO		3.5	3.5		3.8
Dist.		−3.5	−1.7	−1.8	
CO		−0.8	−1.8		−0.9
Dist.		0.8	0.9	0.9	
CO		0.4	0.4		0.4
Dist.		−0.4	−0.2	−0.2	
CO		−0.1	−0.2		−0.1
Dist.		0.1	0.1	0.1	
ΣM	4000	−4000	587.1	−587.1	−293.6

Solution 2



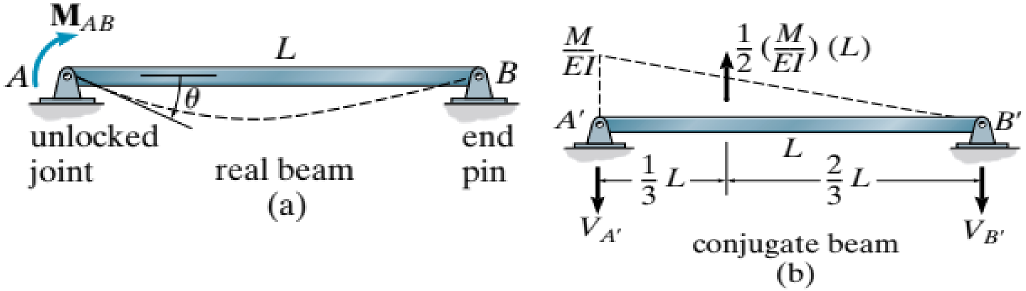
400 lb
400 lb
10 ft
4000 lb · ft
770.6 lb
60 lb/ft
20 ft
429.4 lb
587.1 lb · ft
58.5 lb
15 ft
58.5 lb
293.6 lb · ft
M (lb · ft)
10
949.1
30
293.6
x (ft)
22.8
−587.1
−4000



Moment Distribution for Beams – Stiffness factor modification

- In the previous method, each beam was assumed constrained by fixed supports (locked joints). Thus $SF = 4EI/L$ and $CO = +1/2$. The SF can be modified for simplicity

Member Pin Supported at Far End



$$\circlearrowleft +\Sigma M_{B'} = 0; \quad V'_A(L) - \frac{1}{2}\left(\frac{M}{EI}\right)L\left(\frac{2}{3}L\right) = 0$$

$$V'_A = \theta = \frac{ML}{3EI}$$

or

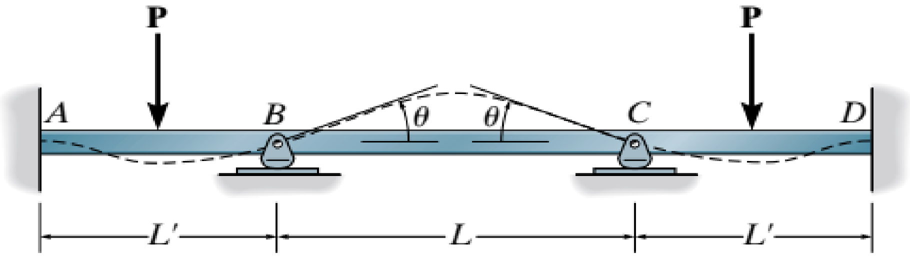
$$M = \frac{3EI}{L}\theta$$

$$K = \frac{3EI}{L}$$

Far End Pinned or Roller Supported

- By comparison, then, if the far end was fixed supported, the stiffness factor $K = 4EI/L$ would have to be modified by 3/4 to model the case of having the far end pin supported.

Symmetric Beam and Loading

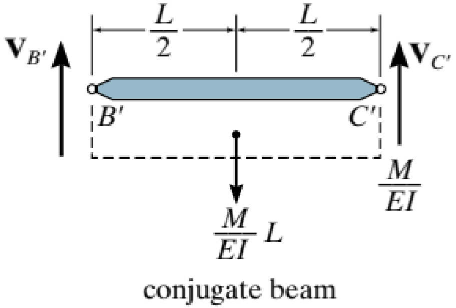


$$\circlearrowleft +\Sigma M_{C'} = 0; \quad -V_{B'}(L) + \frac{M}{EI}(L)\left(\frac{L}{2}\right) = 0$$

$$V_{B'} = \theta = \frac{ML}{2EI}$$

or

$$M = \frac{2EI}{L}\theta$$

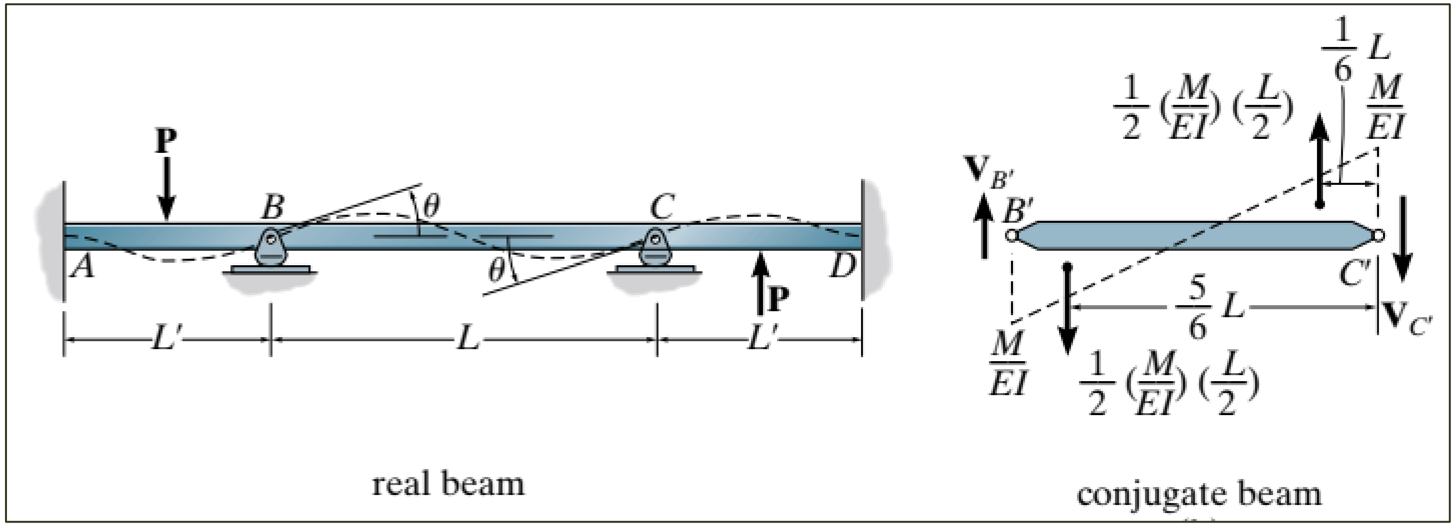


$$K = \frac{2EI}{L}$$

Symmetric Beam and Loading

- The center span's stiffness factor will be one half that usually determined using $K = 4EI/L$.

Symmetric Beam with Antisymmetric Loading



$$K = \frac{6EI}{L}$$

Symmetric Beam with Antisymmetric Loading

$$\circlearrowleft +\Sigma M_{C'} = 0; \quad -V_{B'}(L) + \frac{1}{2}\left(\frac{M}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{5L}{6}\right) - \frac{1}{2}\left(\frac{M}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{6}\right) = 0$$

$$V_{B'} = \theta = \frac{ML}{6EI}$$

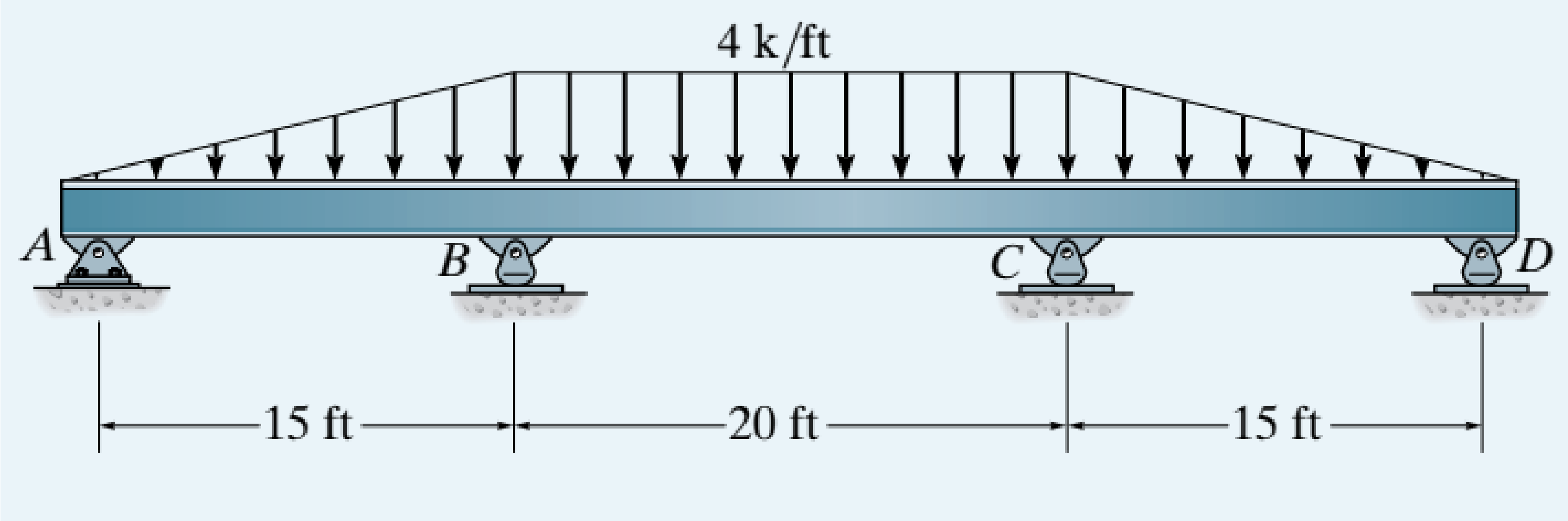
or

$$M = \frac{6EI}{L}\theta$$

- Here the stiffness factor is one and a half times as large as that determined using K = 4EI/L.

Example 3

- Determine the internal moments at the supports for the beam shown in the figure below. EI is constant.



Solution 3

By inspection, the beam and loading are symmetrical. Thus, we will apply $K = 2EI/L$ to compute the stiffness factor of the center span BC and therefore use only the left half of the beam for the analysis. The analysis can be shortened even further by using $K = 3EI/L$ for computing the stiffness factor of segment AB since the far end A is pinned. Furthermore, the distribution of moment at A can be skipped by using the FEM for a triangular loading on a span with one end fixed and the other pinned. Thus,

$$K_{AB} = \frac{3EI}{15} \qquad \text{(using Eq. 12–4)}$$

$$K_{BC} = \frac{2EI}{20} \qquad \text{(using Eq. 12–5)}$$

$$\text{DF}_{AB} = \frac{3EI/15}{3EI/15} = 1$$

$$\text{DF}_{BA} = \frac{3EI/15}{3EI/15 + 2EI/20} = 0.667$$

$$\text{DF}_{BC} = \frac{2EI/20}{3EI/15 + 2EI/20} = 0.333$$

$$(\text{FEM})_{BA} = \frac{wL^2}{15} = \frac{4(15)^2}{15} = 60 \text{ k} \cdot \text{ft}$$

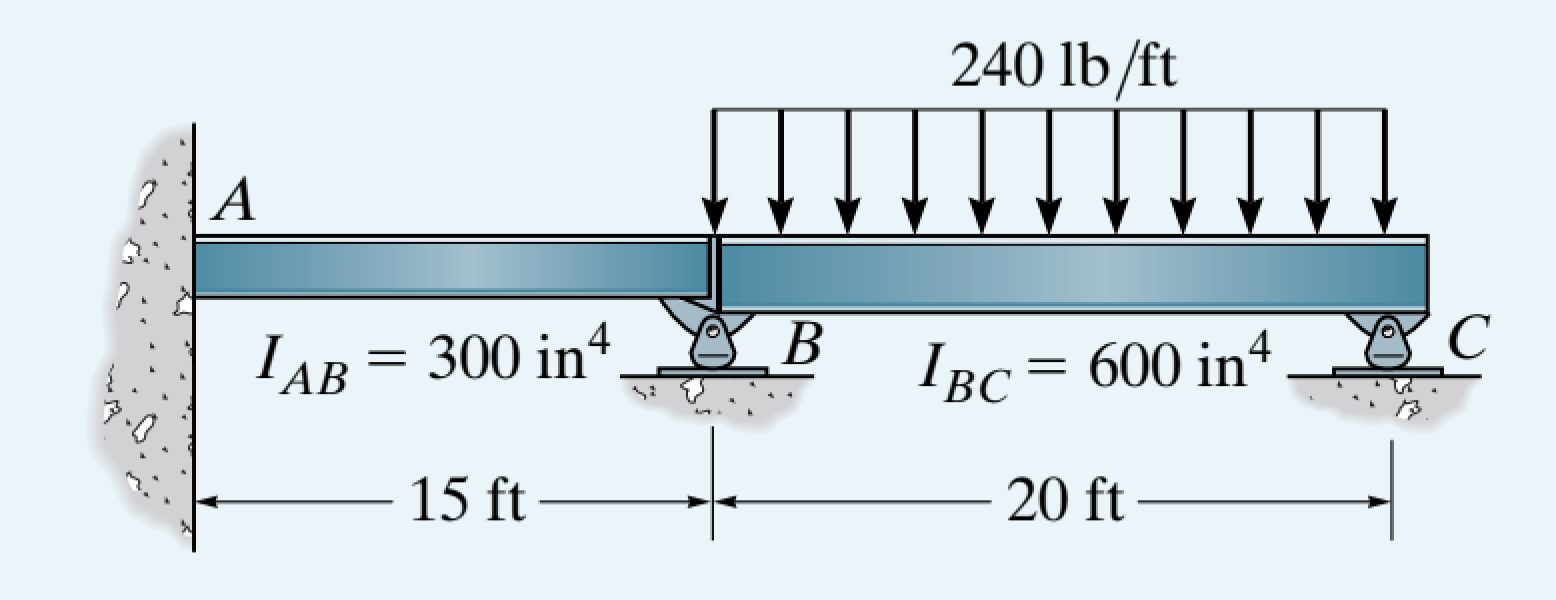
$$(\text{FEM})_{BC} = -\frac{wL^2}{12} = -\frac{4(20)^2}{12} = -133.3 \text{ k} \cdot \text{ft}$$

These data are listed in the table in Fig. 12–13b. Computing the stiffness factors as shown above considerably reduces the analysis, since only joint B must be balanced and carry-overs to joints A and C are not necessary. Obviously, joint C is subjected to the same internal moment of 108.9 k · ft.

Joint	A	B	
Member	AB	BA	BC
DF	1	0.667	0.333
FEM Dist.		60 48.9	−133.3 24.4
ΣM	0	108.9	−108.9

Example 4

- Determine the internal moments at the supports for the beam shown in the figure below. The moment of inertia of the two spans is shown in the figure..



Solution 4

Since the beam is roller supported at its far end C, the stiffness of span BC will be computed on the basis of $K = 3EI/L$. We have

$$K_{AB} = \frac{4EI}{L} = \frac{4E(300)}{15} = 80E$$

$$K_{BC} = \frac{3EI}{L} = \frac{3E(600)}{20} = 90E$$

Thus,

$$\mathrm{DF}_{AB} = \frac{80E}{\infty + 80E} = 0$$

$$\mathrm{DF}_{BA} = \frac{80E}{80E + 90E} = 0.4706$$

$$\mathrm{DF}_{BC} = \frac{90E}{80E + 90E} = 0.5294$$

$$\mathrm{DF}_{CB} = \frac{90E}{90E} = 1$$

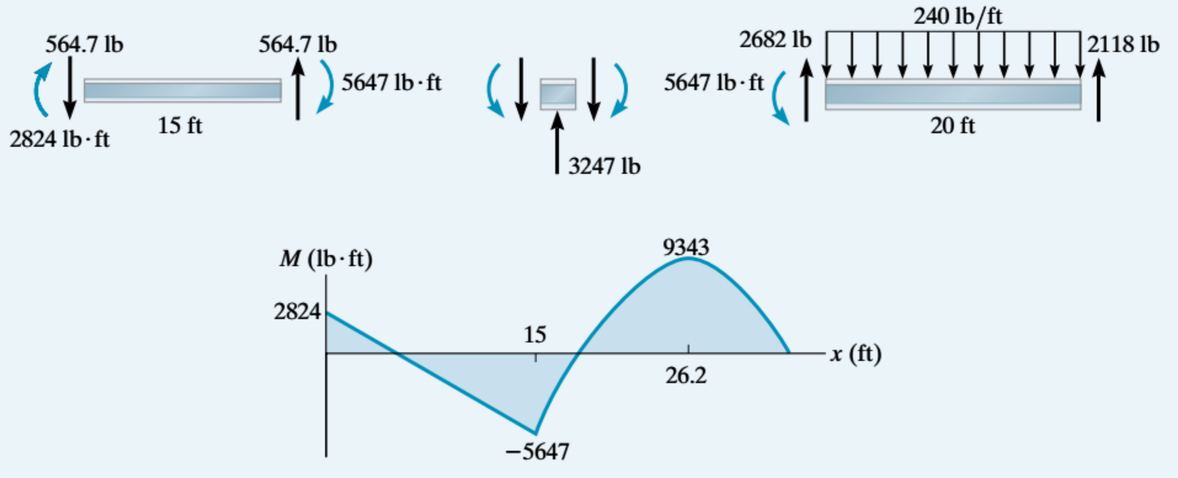
Further simplification of the distribution method for this problem is possible by realizing that a *single* fixed-end moment for the end span BC can be used. Using the right-hand column of the table on the inside back cover for a uniformly loaded span having one side fixed, the other pinned, we have

$$(\mathrm{FEM})_{BC} = -\frac{wL^2}{8} = \frac{-240(20)^2}{8} = -12\,000 \text{ lb}\cdot\text{ft}$$

The foregoing data are entered into the table in Fig. 12–14*b* and the moment distribution is carried out. By comparison with Fig. 12–6*b*, this method considerably simplifies the distribution.

Using the results, the beam's end shears and moment diagrams are shown in Fig. 12–14*c*.

Joint	A	B		C
Member	AB	BA	BC	CB
DF	0	0.4706	0.5294	1
FEM Dist.		 5647.2	−12 000 6352.8	
CO	2823.6			
ΣM	2823.6	5647.2	−5647.2	0

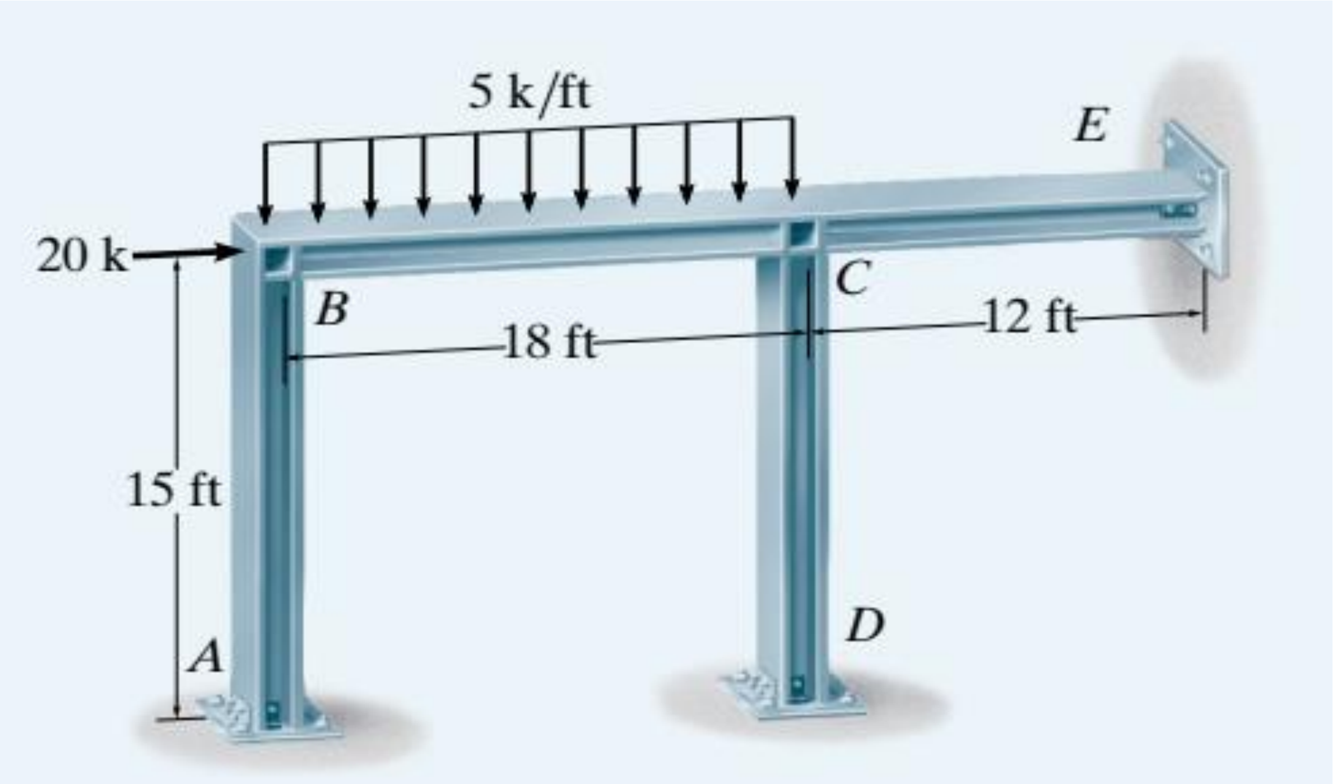


Moment Distribution for Frames – No Sidesway

Application of the moment-distribution method for frames having no sidesway follows the same procedure as that given for beams. To minimize the chance for errors, it is suggested that the analysis be arranged in a tabular form, as in the previous examples. Also, the distribution of moments can be shortened if the stiffness factor of a span can be modified as indicated in the previous section.

Example 5

- Determine the internal moments at the joints of the frame shown in the figure. There is a pin at E and D and a fixed support at A. *EI* is constant.



Solution 5

By inspection, the pin at E will prevent the frame from sidesway. The stiffness factors of CD and CE can be computed using $K = 3EI/L$ since the far ends are pinned. Also, the 20-k load does not contribute a FEM since it is applied at joint B. Thus,

$$K_{AB} = \frac{4EI}{15} \qquad K_{BC} = \frac{4EI}{18} \qquad K_{CD} = \frac{3EI}{15} \qquad K_{CE} = \frac{3EI}{12}$$

$$DF_{AB} = 0$$

$$DF_{BA} = \frac{4EI/15}{4EI/15 + 4EI/18} = 0.545$$

$$DF_{BC} = 1 - 0.545 = 0.455$$

$$DF_{CB} = \frac{4EI/18}{4EI/18 + 3EI/15 + 3EI/12} = 0.330$$

$$DF_{CD} = \frac{3EI/15}{4EI/18 + 3EI/15 + 3EI/12} = 0.298$$

$$DF_{CE} = 1 - 0.330 - 0.298 = 0.372$$

$$DF_{DC} = 1 \qquad DF_{EC} = 1$$

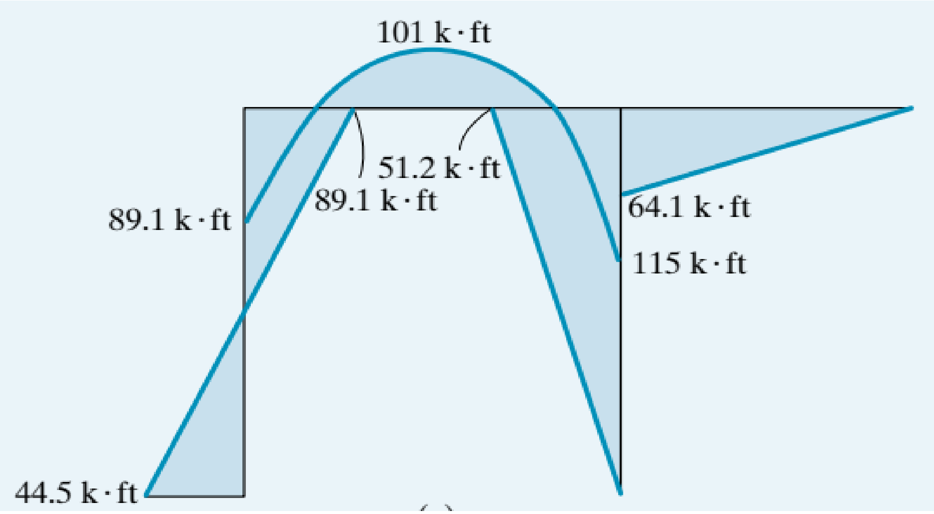
$$(FEM)_{BC} = \frac{-wL^2}{12} = \frac{-5(18)^2}{12} = -135 \text{ k}\cdot\text{ft}$$

$$(FEM)_{CB} = \frac{wL^2}{12} = \frac{5(18)^2}{12} = 135 \text{ k}\cdot\text{ft}$$

The data are shown in the table in Fig. 12–15*b*. Here the distribution of moments successively goes to joints B and C. The final moments are shown on the last line.

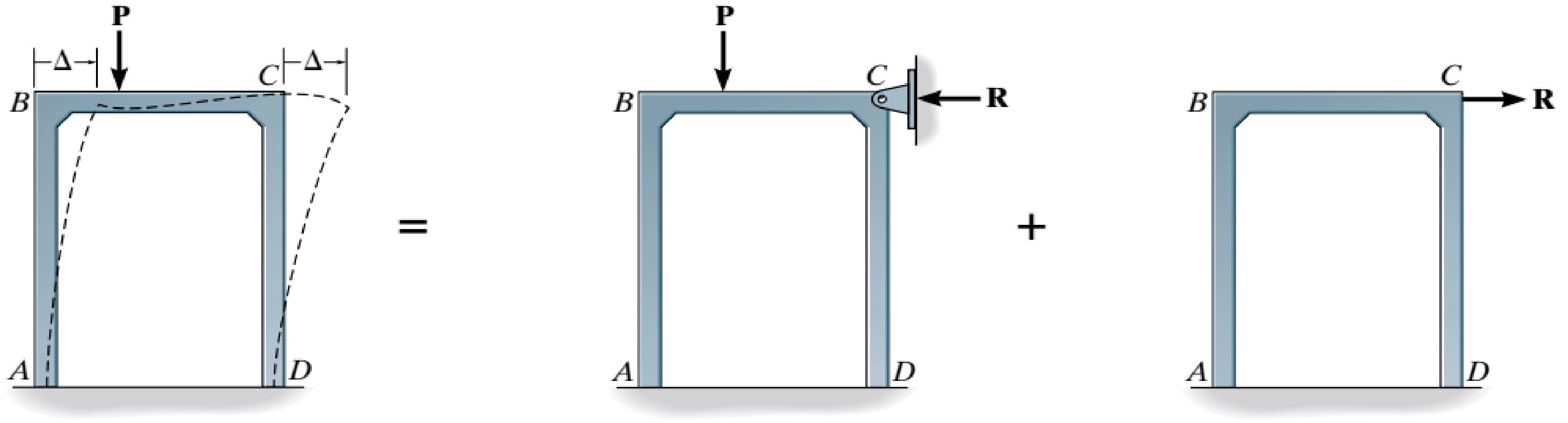
Using these data, the moment diagram for the frame is constructed in Fig. 12–15*c*.

Joint	A	B		C			D	E
Member	AB	BA	BC	CB	CD	CE	DC	EC
DF	0	0.545	0.455	0.330	0.298	0.372	1	1
FEM			−135	135				
Dist.		73.6	61.4	−44.6	−40.2	−50.2		
CO	36.8		−22.3	30.7				
Dist.		12.2	10.1	−10.1	−9.1	−11.5		
CO	6.1		−5.1	5.1				
Dist.		2.8	2.3	−1.7	−1.5	−1.9		
CO	1.4		−0.8	1.2				
Dist.		0.4	0.4	−0.4	−0.4	−0.4		
CO	0.2		−0.2	0.2				
Dist.		0.1	0.1	−0.1	0.0	−0.1		
ΣM	44.5	89.1	−89.1	115	−51.2	−64.1		



THE FOLLOWING IS FOR STUDENT TO STUDY, NOT TO TEACH OR EXAMINE ON

Moment Distribution for Frames – Sidesway



(a)	artificial joint applied (no sidesway) (b)	artificial joint removed (sidesway) (c)

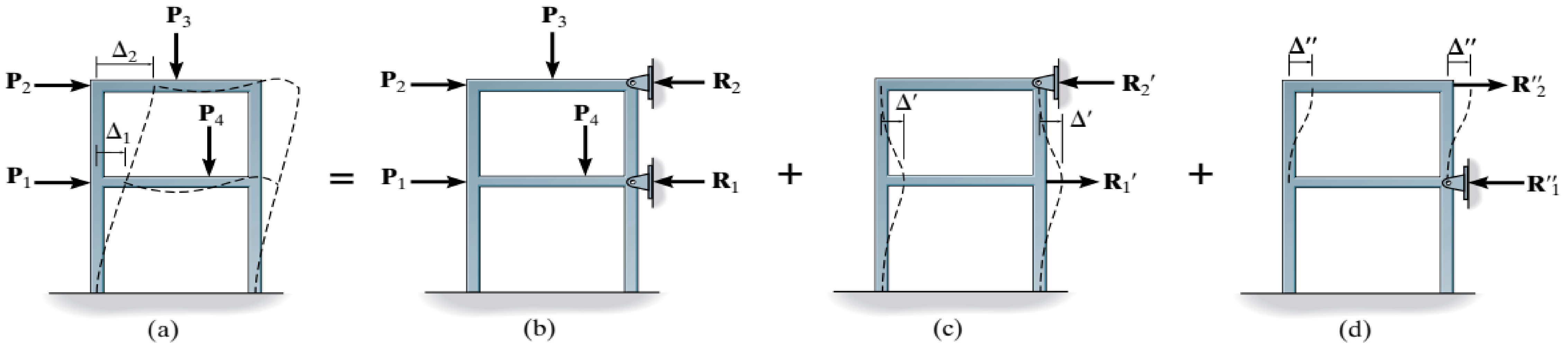
- Here the applied loading **P** will create unequal moments at joints *B* and *C* such that the frame will deflect an amount Δ to the right.
- The frame in figure (b) is first considered held from sidesway by applying an artificial joint support at C.
- Moment distribution is applied and then by statics the restraining force R is determined. The equal, but opposite, restraining force is then applied to the frame

Moment Distribution for Frames – Sidesway

One method for doing this last step requires first *assuming* a numerical value for one of the internal moments, say $\mathbf{M}'_{BA}$. Using moment distribution and statics, the deflection Δ' and external force $\mathbf{R}'$ corresponding to the assumed value of $\mathbf{M}'_{BA}$ can then be determined. Since linear elastic deformations occur, the force $\mathbf{R}'$ develops moments in the frame that are *proportional* to those developed by $\mathbf{R}$. For example, if $\mathbf{M}'_{BA}$ and $\mathbf{R}'$ are known, the moment at B developed by $\mathbf{R}$ will be $M_{BA} = M'_{BA}(R/R')$.

- Addition of the joint moments for both cases, figure b and c, will yield the actual moments in the frame, figure a.

Moment Distribution for Frames – Sidesway



- Consider, for example, the two-story frame shown in figure a. This structure can have two independent joint displacements, since the sidesway Δ_1 of the first story is independent of any displacement Δ_2 of the second story.

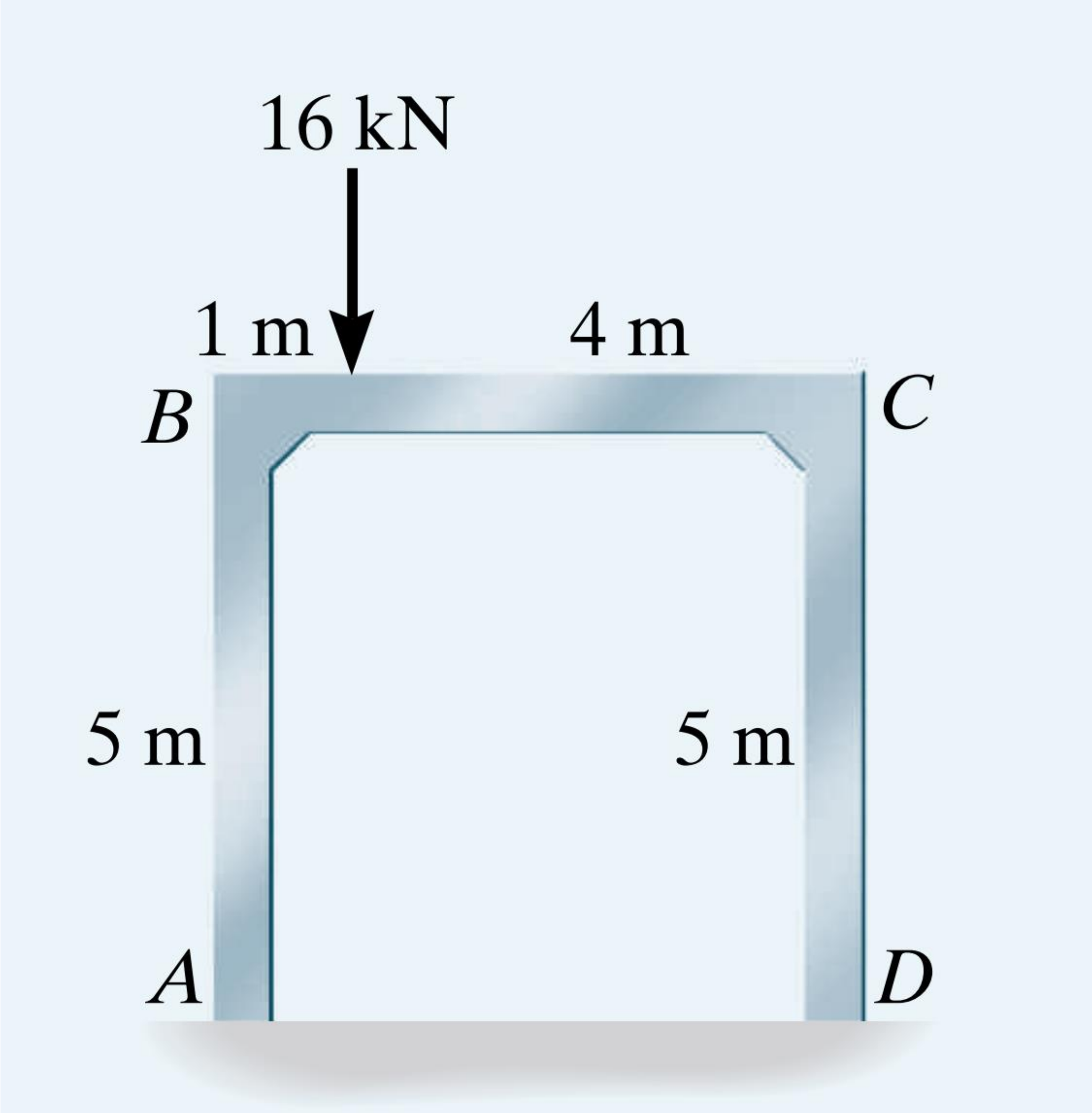
$$R_2 = -C'R_2' + C''R_2''$$

$$R_1 = +C'R_1' - C''R_1''$$

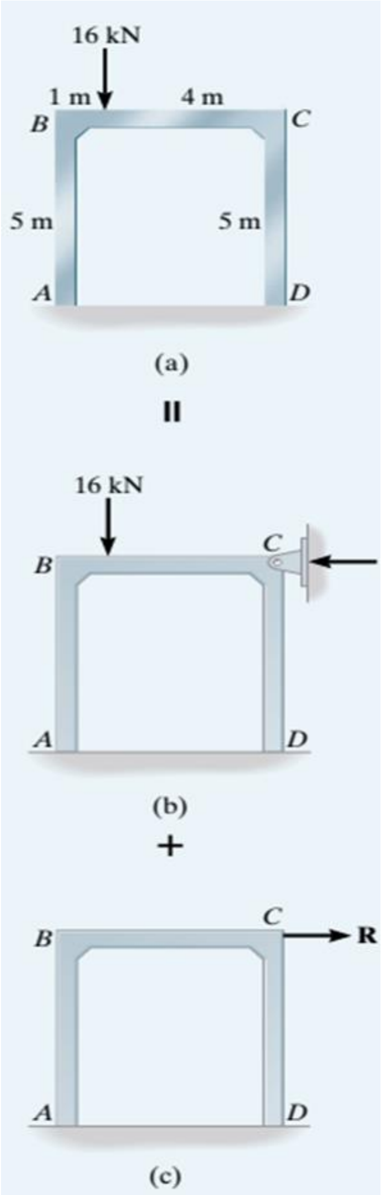
- Simultaneous solution of these equations yields the values of C' and C''. These correction factors are then multiplied by the internal joint moments found from the moment distribution in Fig. c and d. The resultant moments are then found by adding these corrected moments to those obtained for the frame in Fig. b.

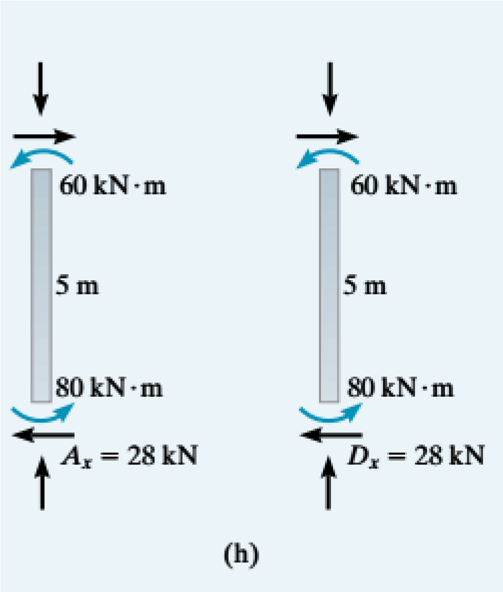
Example 6

- Determine the moments at each joint of the frame shown in the figure. EI is constant



Solution 6





First we consider the frame held from sidesway as shown in Fig. 12–18*b*. We have

$$(\text{FEM})_{BC} = -\frac{16(4)^2(1)}{(5)^2} = -10.24 \text{ kN}\cdot\text{m}$$

$$(\text{FEM})_{CB} = \frac{16(1)^2(4)}{(5)^2} = 2.56 \text{ kN}\cdot\text{m}$$

The stiffness factor of each span is computed on the basis of $4EI/L$ or by using the relative-stiffness factor I/L. The DFs and the moment distribution are shown in the table, Fig. 12–18*d*. Using these results, the equations of equilibrium are applied to the free-body diagrams of the columns in order to determine $\mathbf{A}_x$ and $\mathbf{D}_x$ Fig. 12–18*e*. From the free-body diagram of the entire frame (not shown) the joint restraint **R** in Fig. 12–18*b* has a magnitude of

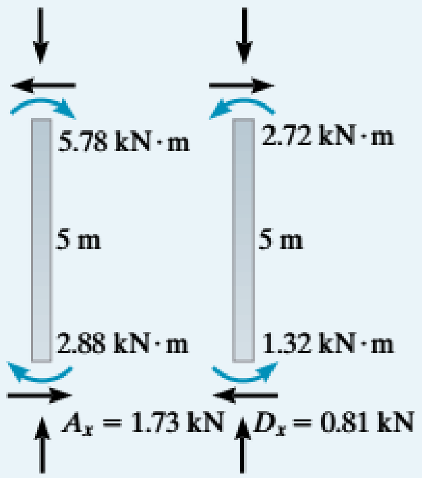
$$\Sigma F_x = 0; \qquad R = 1.73 \text{ kN} - 0.81 \text{ kN} = 0.92 \text{ kN}$$

An equal but opposite value of $R = 0.92$ kN must now be applied to the frame at C and the internal moments computed, Fig. 12–18*c*. To solve the problem of computing these moments, we will assume a force $\mathbf{R}'$ is applied at C, causing the frame to deflect Δ' as shown in Fig. 12–18*f*. Here the joints at B and C are *temporarily restrained from rotating*, and as a result the fixed-end moments at the ends of the columns are determined from the formula for deflection found on the inside back cover, that is,

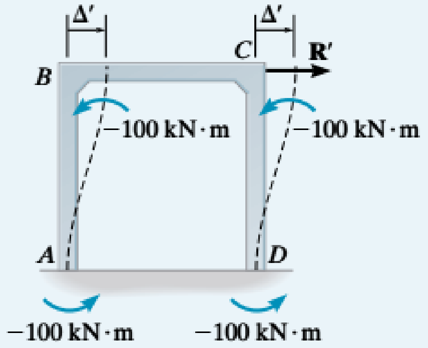
Solution 6

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	0	0.5	0.5	0.5	0.5	0
FEM			−10.24	2.56		
Dist.		5.12	5.12	−1.28	−1.28	
CO	2.56		−0.64	2.56		−0.64
Dist.		0.32	0.32	−1.28	−1.28	
CO	0.16		−0.64	0.16		−0.64
Dist.		0.32	0.32	−0.08	−0.08	
CO	0.16		−0.04	0.16		−0.04
Dist.		0.02	0.02	−0.08	−0.08	
ΣM	2.88	5.78	−5.78	2.72	−2.72	−1.32

(d)



(e)



(f)

$$M = \frac{6EI\Delta}{L^2}$$

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	0	0.5	0.5	0.5	0.5	0
FEM	−100	−100			−100	−100
Dist.		50	50	50	50	
CO	25		25	25		25
Dist.		−12.5	−12.5	−12.5	−12.5	
CO	−6.25		−6.25	−6.25		−6.25
Dist.		3.125	3.125	3.125	3.125	
CO	1.56		1.56	1.56		1.56
Dist.		−0.78	−0.78	−0.78	−0.78	
CO	−0.39		−0.39	−0.39		−0.39
Dist.		0.195	0.195	0.195	0.195	
ΣM	−80.00	−60.00	60.00	60.00	−60.00	−80.00

(g)

Since *both* B and C happen to be displaced the same amount Δ', and AB and DC have the *same* E, I, and L, the FEM in AB will be the *same* as that in DC. As shown in Fig. 12–18*f*, we will arbitrarily *assume* this fixed-end moment to be

$$(FEM)_{AB} = (FEM)_{BA} = (FEM)_{CD} = (FEM)_{DC} = -100 \text{ kN}\cdot\text{m}$$

A *negative sign* is necessary since the moment must act *counterclockwise* on the column for deflection Δ' to the right. The value of $\mathbf{R}'$ associated with this −100 kN·m moment can now be determined. The moment distribution of the FEMs is shown in Fig. 12–18*g*. From equilibrium, the horizontal reactions at A and D are calculated, Fig. 12–18*h*. Thus, for the entire frame we require

$$\Sigma F_x = 0; \qquad R' = 28 + 28 = 56.0 \text{ kN}$$

Hence, $R' = 56.0$ kN creates the moments tabulated in Fig. 12–18*g*. Corresponding moments caused by $R = 0.92$ kN can be determined by proportion. Therefore, the resultant moment in the frame, Fig. 12–18*a*, is equal to the *sum* of those calculated for the frame in Fig. 12–18*b* plus the proportionate amount of those for the frame in Fig. 12–18*c*. We have

$$M_{AB} = 2.88 + \tfrac{0.92}{56.0}(-80) = 1.57 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$

$$M_{BA} = 5.78 + \tfrac{0.92}{56.0}(-60) = 4.79 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$

$$M_{BC} = -5.78 + \tfrac{0.92}{56.0}(60) = -4.79 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$

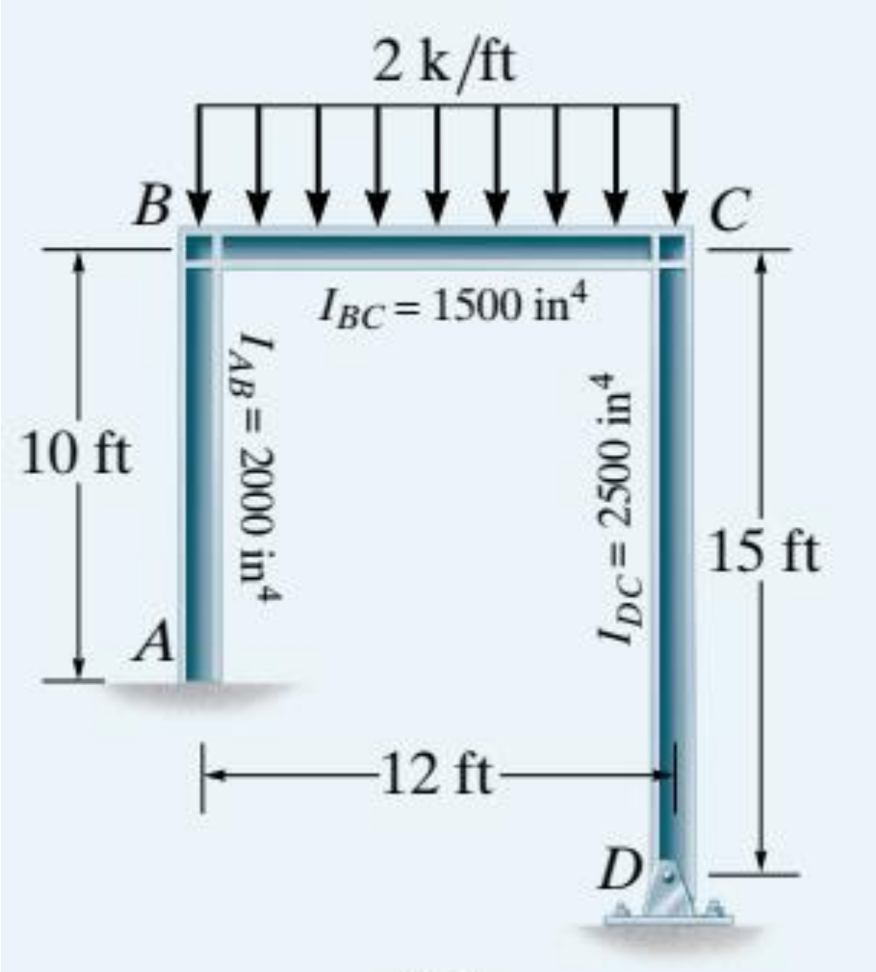
$$M_{CB} = 2.72 + \tfrac{0.92}{56.0}(60) = 3.71 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$

$$M_{CD} = -2.72 + \tfrac{0.92}{56.0}(-60) = -3.71 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$

$$M_{DC} = -1.32 + \tfrac{0.92}{56.0}(-80) = -2.63 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$

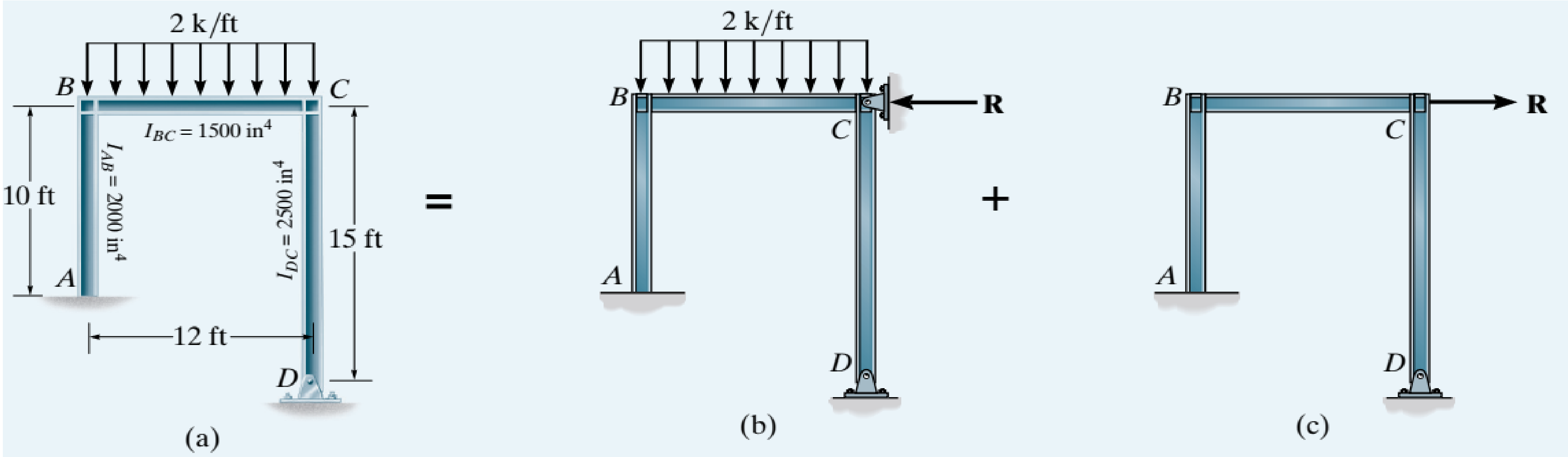
Example 7

- Determine the moments at each joint of the frame shown in the figure a. The moment of inertia of each member is indicated in the figure



(a)

Solution 7

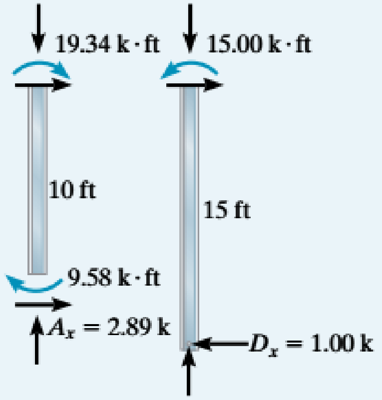


The frame is first held from sidesway as shown in Fig. 12–19*b*. The internal moments are computed at the joints as indicated in Fig. 12–19*d*. Here the stiffness factor of *CD* was computed using $3EI/L$ since there is a pin at *D*. Calculation of the horizontal reactions at *A* and *D* is shown in Fig. 12–19*e*. Thus, for the entire frame,

$$\Sigma F_x = 0; \qquad R = 2.89 - 1.00 = 1.89\ \text{k}$$

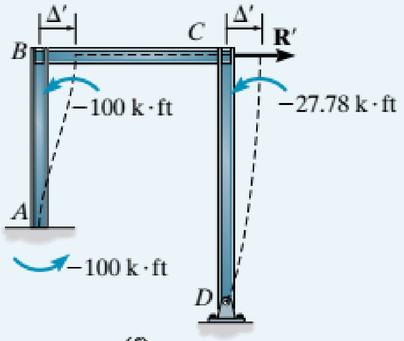
Joint	*A*	*B*		*C*		*D*
Member	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
DF	0	0.615	0.385	0.5	0.5	1
FEM			−24	24		
Dist.		14.76	9.24	−12	−12	
CO	7.38		−6	4.62		
Dist.		3.69	2.31	−2.31	−2.31	
CO	1.84		−1.16	1.16		
Dist.		0.713	0.447	−0.58	−0.58	
CO	0.357		−0.29	0.224		
Dist.		0.18	0.11	−0.11	−0.11	
ΣM	9.58	19.34	−19.34	15.00	−15.00	0

(d)



(e)

Solution 7



(f)

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	0	0.615	0.385	0.5	0.5	1
FEM	−100	−100			−27.78	
Dist.		61.5	38.5	13.89	13.89	
CO	30.75		6.94	19.25		
Dist.		−4.27	−2.67	−9.625	−9.625	
CO	−2.14		−4.81	−1.34		
Dist.		2.96	1.85	0.67	0.67	
CO	1.48		0.33	0.92		
Dist.		−0.20	−0.13	−0.46	−0.46	
ΣM	−69.91	−40.01	40.01	23.31	−23.31	0

(g)

The opposite force is now applied to the frame as shown in Fig. 12–19*c*. As in the previous example, we will consider a force **R′** acting as shown in Fig. 12–19*f*. As a result, joints B and C are displaced by the same amount Δ'. The fixed-end moments for BA are computed from

$$(\text{FEM})_{AB} = (\text{FEM})_{BA} = -\frac{6EI\Delta}{L^2} = -\frac{6E(2000)\Delta'}{(10)^2}$$

However, from the table on the inside back cover, for CD we have

$$(\text{FEM})_{CD} = -\frac{3EI\Delta}{L^2} = -\frac{3E(2500)\Delta'}{(15)^2}$$

Assuming the FEM for AB is −100 k · ft as shown in Fig. 12–19*f*, the *corresponding* FEM at C, causing the *same* Δ', is found by comparison, i.e.,

$$\Delta' = -\frac{(-100)(10)^2}{6E(2000)} = -\frac{(\text{FEM})_{CD}(15)^2}{3E(2500)}$$

$$(\text{FEM})_{CD} = -27.78 \text{ k}\cdot\text{ft}$$

Moment distribution for these FEMs is tabulated in Fig. 12–19*g*. Computation of the horizontal reactions at A and D is shown in Fig. 12–19*h*. Thus, for the entire frame,

$$\Sigma F_x = 0; \qquad R' = 11.0 + 1.55 = 12.55 \text{ k}$$

The resultant moments in the frame are therefore

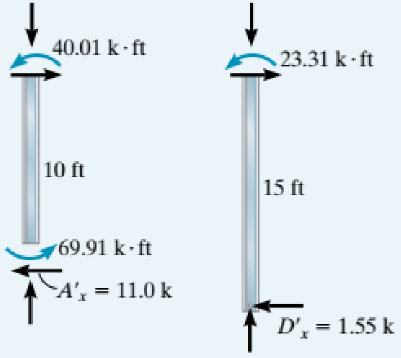
$$M_{AB} = 9.58 + \left(\tfrac{1.89}{12.55}\right)(-69.91) = -0.948 \text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{BA} = 19.34 + \left(\tfrac{1.89}{12.55}\right)(-40.01) = 13.3 \text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

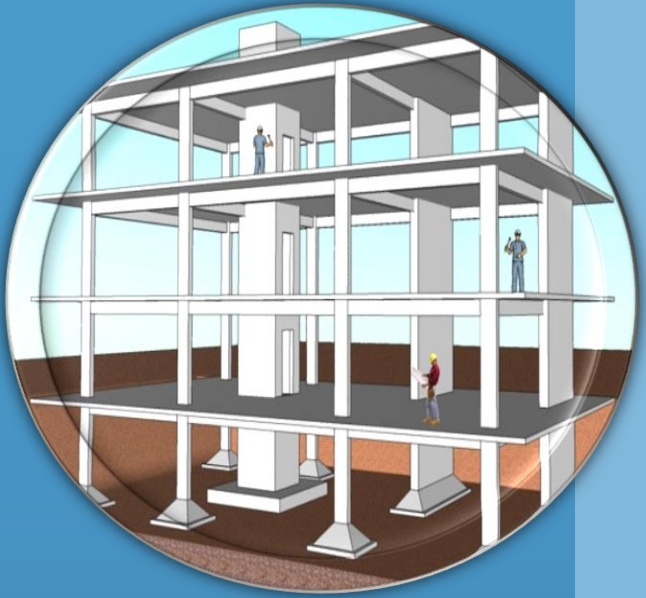
$$M_{BC} = -19.34 + \left(\tfrac{1.89}{12.55}\right)(40.01) = -13.3 \text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{CB} = 15.00 + \left(\tfrac{1.89}{12.55}\right)(23.31) = 18.5 \text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{CD} = -15.00 + \left(\tfrac{1.89}{12.55}\right)(-23.31) = -18.5 \text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$



(h)



Thank You!