SCHOOL OF ENGINEERING

DEPARTMENT OF CIVIL AND ENVIRONMENTAL ENGINEERING

CEE 3211– MECHANICS OF MATERIALS

LECTURE 4 – STATICALLY INDETERMINACY

Static Indeterminacy

- Structures for which internal forces and reactions cannot be determined from statics alone are said to be *statically indeterminate*.

- A structure will be statically indeterminate whenever it is held by more supports than are required to maintain its equilibrium.

- Redundant reactions are replaced with unknown loads which along with the other loads must produce compatible deformations.

- Deformations due to actual loads (δ_L) and redundant reactions (δ_R) are determined separately and then added or *superposed*.

$$\delta = \delta_L + \delta_R = 0$$

STATIC INDETERMINACY – SUPERPOSITION

Superposition Method

- We observe that a structure is statically indeterminate whenever it is held by more supports than are required to maintain its equilibrium.
- This results in more unknown reactions than available equilibrium equations.
- It is often found convenient to designate one of the reactions as *redundant* and to eliminate the corresponding support.

STATIC INDETERMINACY – SUPERPOSITION

- Since the stated conditions of the problem cannot be arbitrarily changed, the redundant reaction must be maintained in the solution

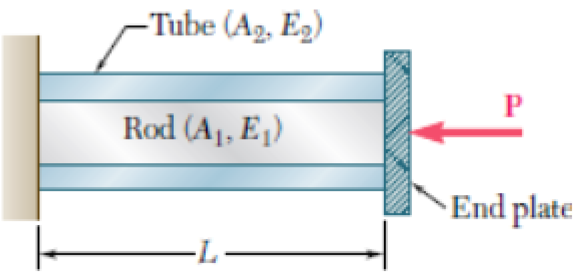
- But it will be treated as an *unknown load* that, together with the other loads, must produce deformations that are compatible with the original constraints.

- The actual solution of the problem is carried out by considering separately the deformations caused by the given loads and by the redundant reaction, and by adding—or *superposing*—the results obtained

Compatibility Conditions

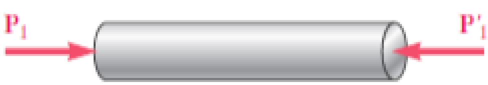
- When the force equilibrium condition alone cannot determine the solution, the structural member is called **statically indeterminate.**
- In this case, compatibility conditions at the constraint locations shall be used to obtain the solution.

Illustration of statically indeterminacy

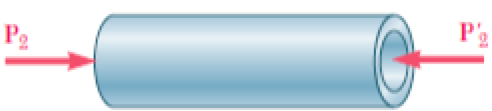


(*a*)

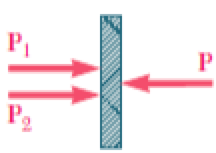
A rod of length L, cross-sectional area A_1, and modulus of elasticity E_1, has been placed inside a tube of the same length L, but of cross-sectional area A_2 and modulus of elasticity E_2 . What is the deformation of the rod and tube when a force **P** is exerted on a rigid end plate as shown?



(*b*)



(*c*)



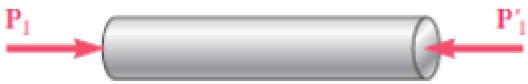
(*d*)

Denoting by P_1 and P_2, respectively, the axial forces in the rod and in the tube, we draw free-body diagrams of all three elements (Fig.*b*,*c*, *d*). Only the last of the diagrams yields any significant information, namely:

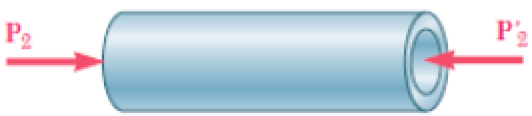
$$P_1 + P_2 = P \longrightarrow \text{(eqn a)}$$

Clearly, one equation is not sufficient to determine the two unknown internal forces P_1 and P_2. The problem is statically indeterminate. However, the geometry of the problem shows that the deformations δ_1 and δ_2 of the rod and tube must be equal.

Illustration of statically indeterminacy



(b)



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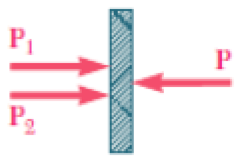


(d)

$$\delta_1 = \frac{P_1 L_1}{A_1 E_1} \qquad \delta_2 = \frac{P_2 L_2}{A_2 E_2}$$

The 2 deformations are equal; hence equating both equations, yields:

$$\frac{P_1}{A_1 E_1} = \frac{P_2}{A_2 E_2}$$ (eqn b)

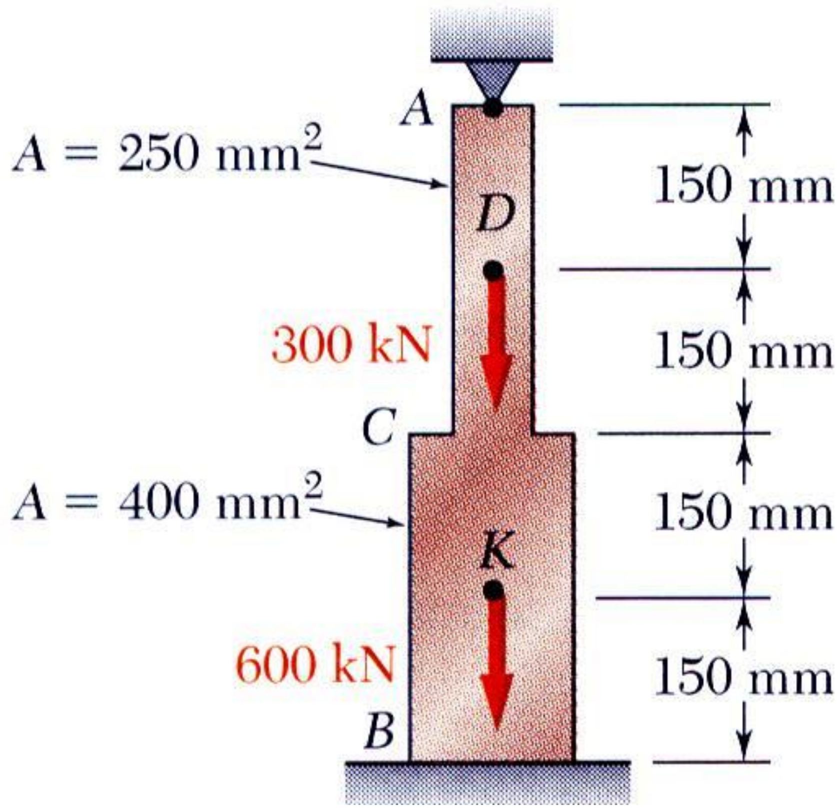
Solving Eqn (a) and (b) for P_1 and P_2

$$P_1 = \frac{A_1 E_1 P}{A_1 E_1 + A_2 E_2}$$

$$P_2 = \frac{A_2 E_2 P}{A_1 E_1 + A2 E_2}$$

Example 1

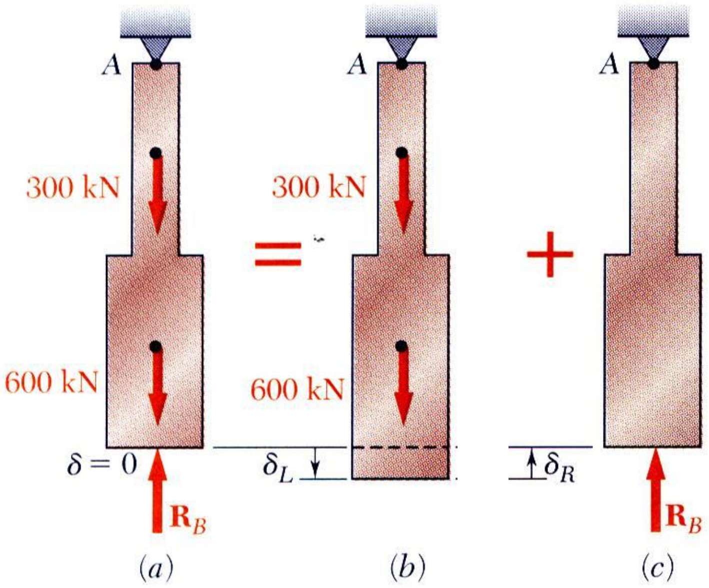
Determine the reactions at A and B for the steel bar and loading shown, assuming a close fit at both supports before the loads are applied.



Example 1

SOLUTION:

- Consider the reaction at *B* as redundant, release the bar from that support, and solve for the displacement at *B* due to the applied loads.



- Solve for the displacement at *B* due to the redundant reaction at *B.*

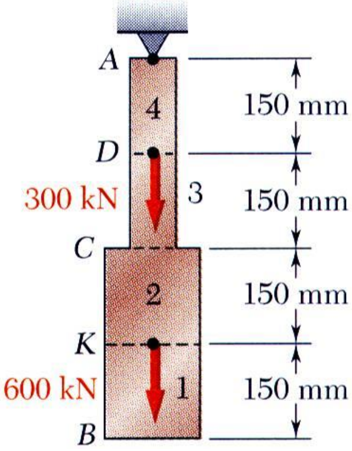
- Require that the displacements due to the loads and due to the redundant reaction be compatible, i.e., require that their sum be zero.

- Solve for the reaction at *A* due to applied loads and the reaction found at *B*.

Example 1 Continued

SOLUTION:

- Solve for the displacement at *B*, δ_L due to the applied loads with the redundant constraint released,



$$P_1 = 0$$
$$P_2 = P_3 = 600 \times 10^3 \mathrm{N}$$
$$P_4 = 900 \times 10^3 \mathrm{N}$$

$$A_1 = A_2 = 400 \times 10^{-6} \mathrm{m}^2$$
$$A_3 = A_4 = 250 \times 10^{-6} \mathrm{m}^2$$
$$L_1 = L_2 = L_3 = L_4 = 0.150 \text{ m}$$

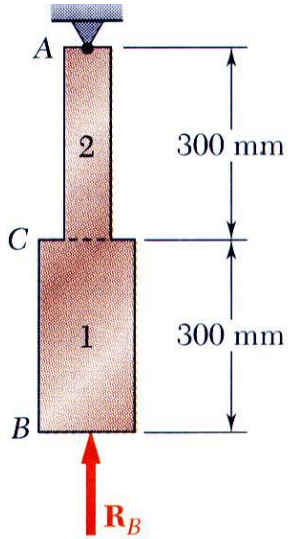
Example 1 Continued

$$\delta_L = \sum_i \frac{P_i L_i}{A_i E_i}$$

$$\delta_L = \sum_{i=1}^{4} \frac{P_i L_i}{A_i E} = \left(0 + \frac{600 \times 10^3\,\mathrm{N}}{400 \times 10^{-6}\,\mathrm{m}^2} + \frac{600 \times 10^3\,\mathrm{N}}{250 \times 10^{-6}\,\mathrm{m}^2} + \frac{900 \times 10^3\,\mathrm{N}}{250 \times 10^{-6}\,\mathrm{m}^2}\right)\frac{0.150\,\mathrm{m}}{E}$$

$$\boldsymbol{\delta_L = \frac{(1.125 \times 10^9)}{E}} \longrightarrow \text{(a)}$$

Example 1 Continued



- Solve for the displacement, δ_R at B due to the redundant constraint, we divide the bar into two portions,

$$\delta_R = \frac{P_1 L_1}{A_1 E} + \frac{P_2 L_2}{A_2 E}$$

$$P_1 = P_2 = -R_B$$
$$A_1 = 400 \times 10^{-6} \mathrm{m}^2$$
$$A_2 = 250 \times 10^{-6} \mathrm{m}^2$$
$$L_1 = L_2 = 0.300 \text{ m}$$

Example 1 Continued

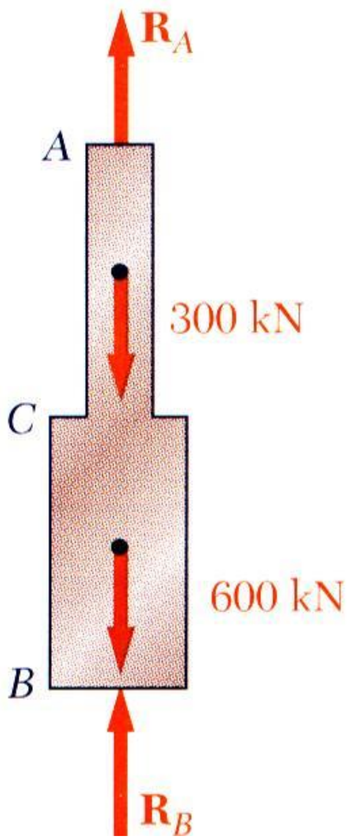
$$\delta_R = \frac{P_1 L_1}{A_1 E} + \frac{P_2 L_2}{A_2 E}$$

$$\delta_R = -\frac{(1.95 \times 10^3)}{E} R_B \longrightarrow \text{(b)}$$

Total deformation:

$$\delta = \delta_L + \delta_R = 0$$

Example 1 Continued



- Expressing that the displacements due to the loads and due to the redundant reaction be compatible,

$$\delta = \delta_L + \delta_R = 0$$

$$\delta = \frac{1.125\times10^9}{E} - \frac{\left(1.95\times10^3\right)R_B}{E} = 0$$

$$R_B = 577\times10^3\,\mathrm{N} = 577\,\mathrm{kN}$$

- Find the reaction at *A* due to the loads and the reaction at *B*

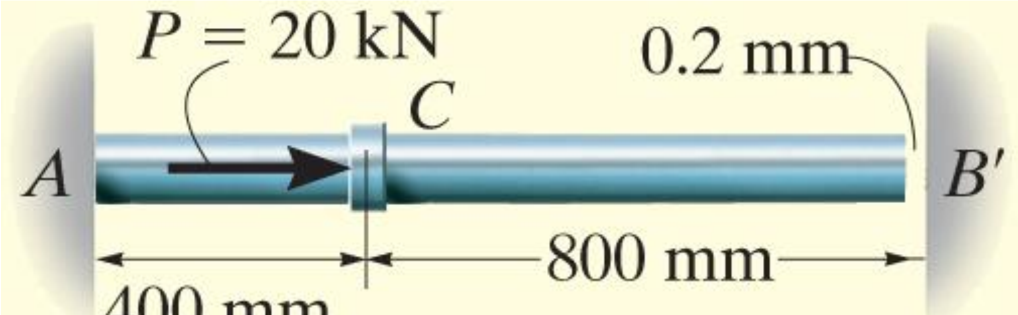
$$\sum F_y = 0 = R_A - 300\,\mathrm{kN} - 600\,\mathrm{kN} + 577\,\mathrm{kN}$$

$$R_A = 323\,\mathrm{kN}$$

$$\boxed{\begin{aligned} R_A &= 323\mathrm{kN} \\ R_B &= 577\mathrm{kN} \end{aligned}}$$

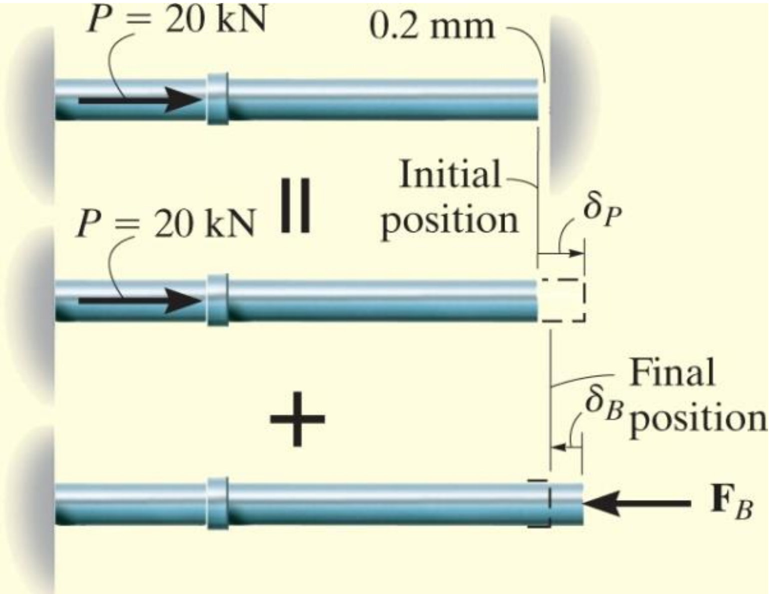
Example 2 (1 of 3)

The A-36 steel rod shown below has a diameter of 10 mm. It is fixed to the wall at *A*, and before it is loaded there is a gap between the wall at B' and the rod of 0.2 mm. Determine the reactions at *A* and Neglect the size of the collar at *C*. Take Est = 200GPa.



(a)

Example 2 (2 of 3)



(b)

Solutions

- Using the principle of superposition,

$$\left(\xrightarrow{+}\right) \quad 0.0002 = \delta_P - \delta_B \qquad (1)$$

- From Equation 4-2,

$$\delta_P = \frac{PL_{AC}}{AE} = \frac{\left[20\left(10^3\right)\right](0.4)}{\pi(0.005)^2\left[200\left(10^9\right)\right]} = 0.5093\left(10^{-3}\right)$$

$$\delta_B = \frac{F_B L_{AB}}{AE} = \frac{F_B(1.2)}{\pi(0.005)^2\left[200\left(10^9\right)\right]} = 76.3944\left(10^{-9}\right)F_B$$

- Substituting into Equation 1, we get

$$0.0002 = 0.5093\left(10^{-3}\right) - 76.3944\left(10^{-9}\right)F_B$$

$$F_B = 4.05\left(10^3\right) = 4.05\text{ kN} \quad \text{(Ans)}$$

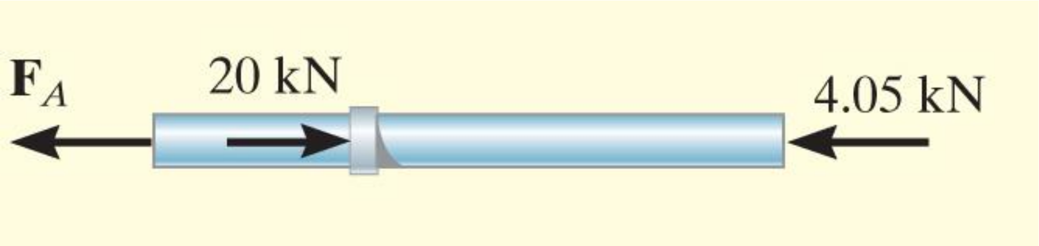
Example 2 (3 of 3)

- From the free-body diagram,

$$(+\rightarrow) \quad \sum F_x = 0$$

$$-F_A + 20 - 4.05 = 0$$

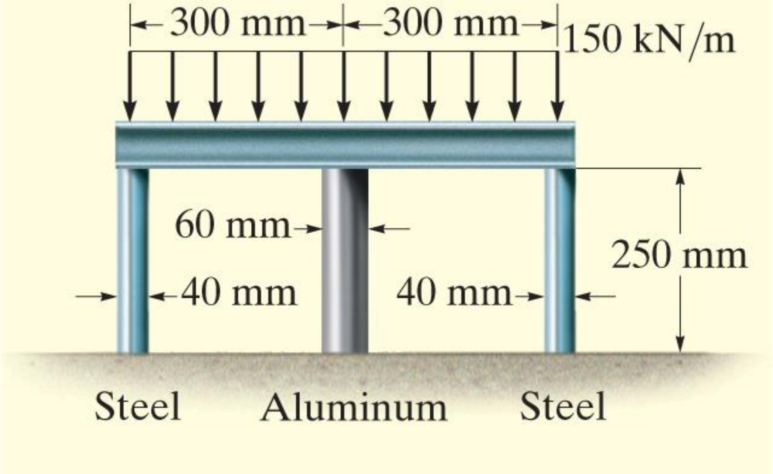
$$F_A = 16.0\ \text{kN} \quad \text{(Ans)}$$



(c)

Example 3 (1 of 3)

The rigid bar is fixed to the top of the three posts made of A-36 steel and 2014-T6 aluminum. The posts each have a length of 250 mm when no load is applied to the bar, and the temperature is T1 = 20° C. Determine the force supported by each post if the bar is subjected to a uniform distributed load of 150 kN/m and the temperature is raised to T2 = 20° C.



(a)

Example 3 (2 of 3)

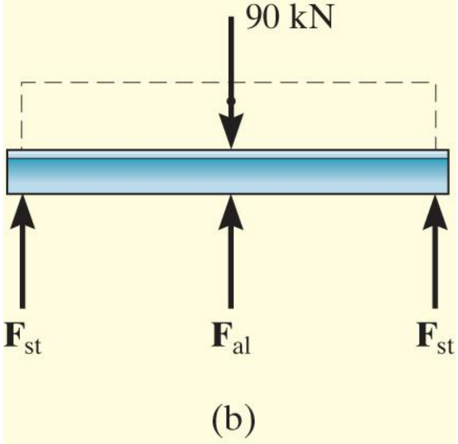
Solutions

- From the free-body diagram we have

$$+\uparrow \sum F_y = 0; \quad 2F_{st} + F_{al} - 90\left(10^3\right) = 0 \quad (1)$$

- The top of each post is displaced by an equal amount and hence,

$$\left(+\downarrow\right) \quad \delta_{st} = \delta_{al} \quad (2)$$



- Final position of the top of each post is equal to its displacement caused by the temperature increase and internal axial compressive force.

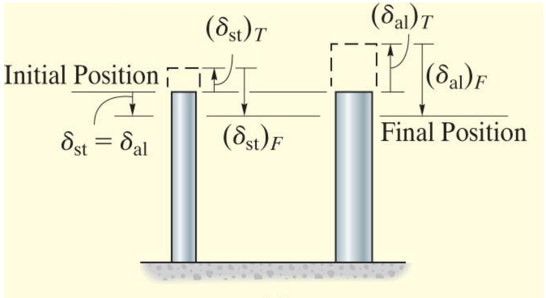
$$\left(+\downarrow\right) \quad \delta_{st} = -\left(\delta_{st}\right)_T + \left(\delta_{st}\right)_F$$

$$\left(+\downarrow\right) \quad \delta_{al} = -\left(\delta_{al}\right)_T + \left(\delta_{al}\right)_F$$

Example 3 (3 of 3)

- Applying Equation 2 gives

$$-(\delta_{st})_T + (\delta_{st})_F = -(\delta_{st})_T + (\delta_{al})_F$$



(c)

- With reference from the material properties, we have

$$-\left[12\left(10^{-6}\right)\right](80-20)(0.25) + \frac{F_{st}(0.25)}{\pi(0.02)^2\left[200\left(10^9\right)\right]} = -\left[23\left(10^{-6}\right)\right](80-20)(0.25) + \frac{F_{al}(0.25)}{\pi(0.03)^2\left[73.1\left(10^9\right)\right]}$$

$$F_{st} = 1.216\ F_{al} - 165.9\left(10^3\right) \qquad (3)$$

- Solving Equations 1 and 3 simultaneously yields

$$F_{st} = -16.4\ \text{kN} \ \text{ and } \ F_{al} = 123\ \text{kN} \quad \text{(Ans)}$$