SCHOOL OF ENGINEERING

DEPARTMENT OF CIVIL AND ENVIRONMENTAL ENGINEERING

# CEE 3211– MECHANICS OF MATERIALS

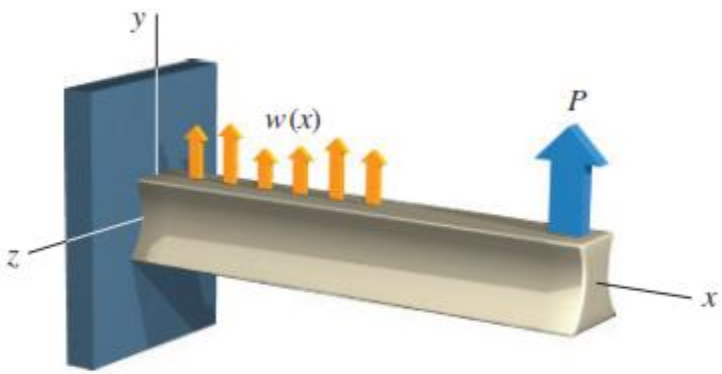
# LECTURE 10 – THEORY OF BENDING STRESS

# Pure Bending in Beams

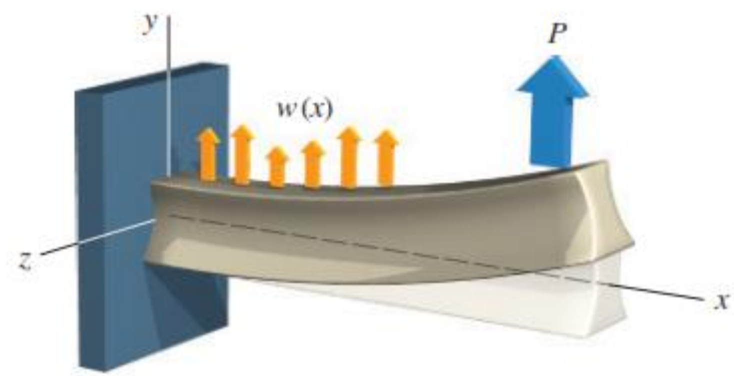
# Bending in Beams

The Beam is the most common type of structural member is the beam. In actual structures and machines, beams can be found in a wide variety of sizes

Beams are usually long, straight, prismatic members that support transverse loads, which are loads that act perpendicular to the longitudinal axis of the member. The applied load causes the beam to deform

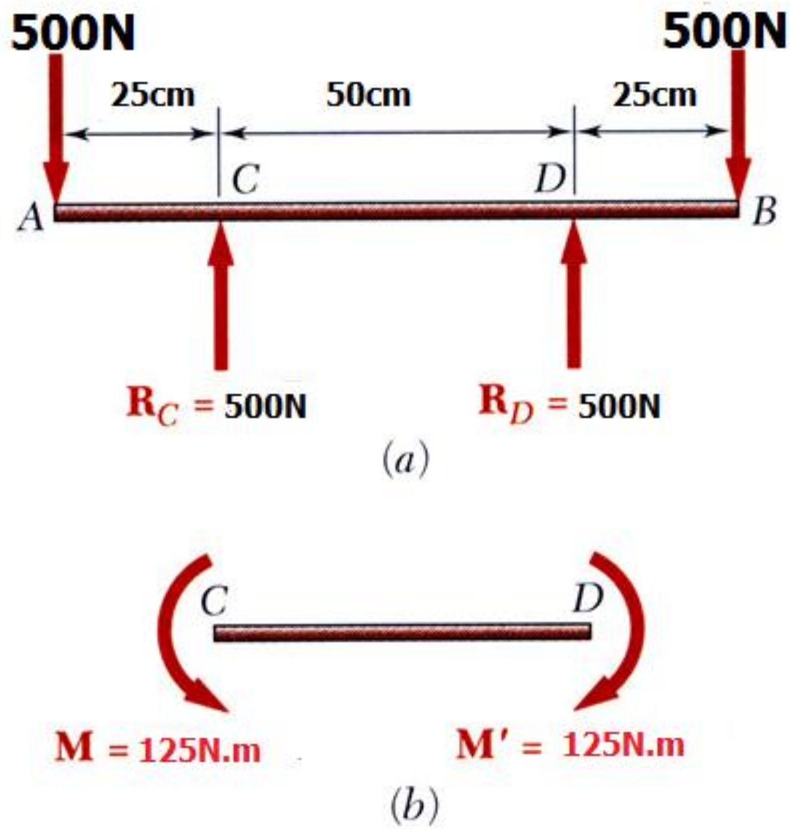



Beam with applied loads

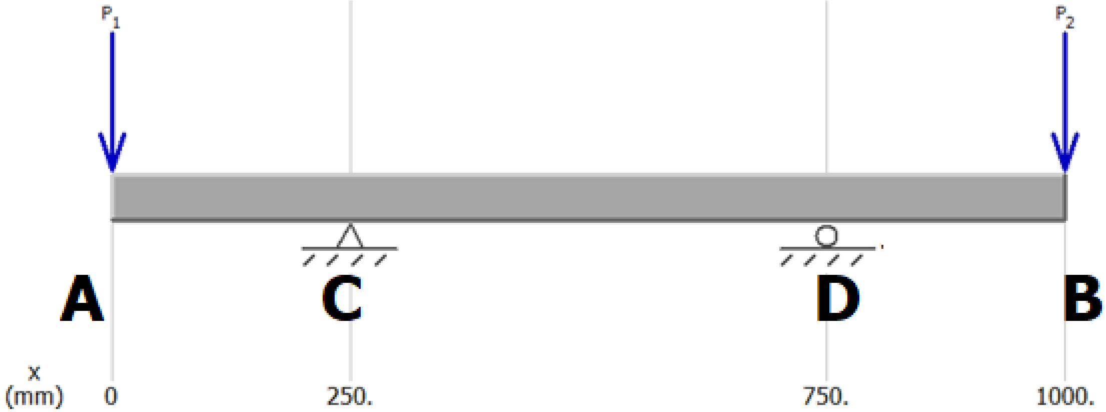



Deformed due to bending

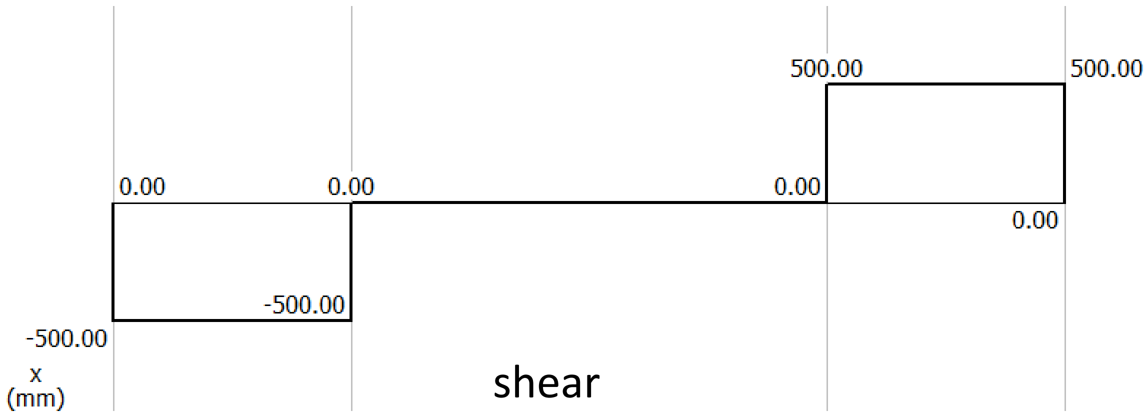
# Pure Bending in Beams



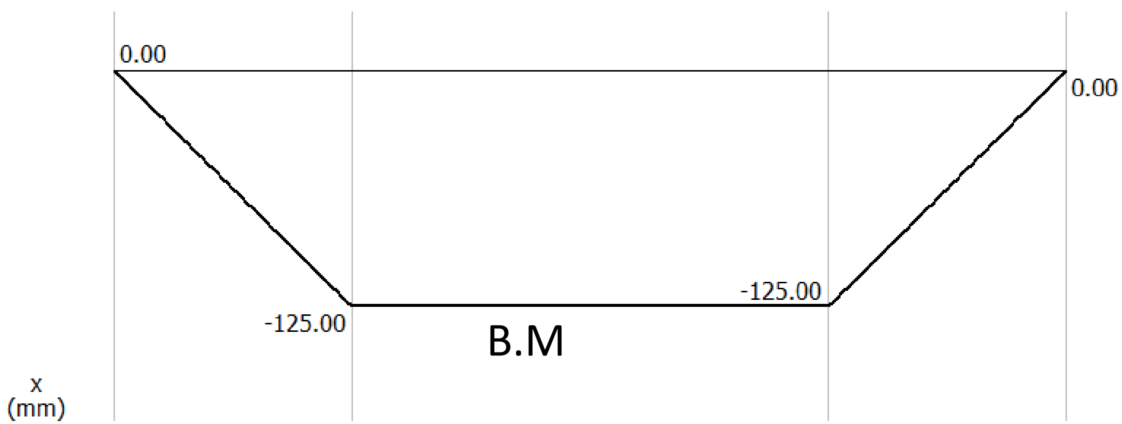

*Pure Bending*: Prismatic members subjected to equal and opposite couples acting in the same longitudinal plane




**Pure bending** refers to the flexure of a beam in response to constant (i.e., equal) bending moments.

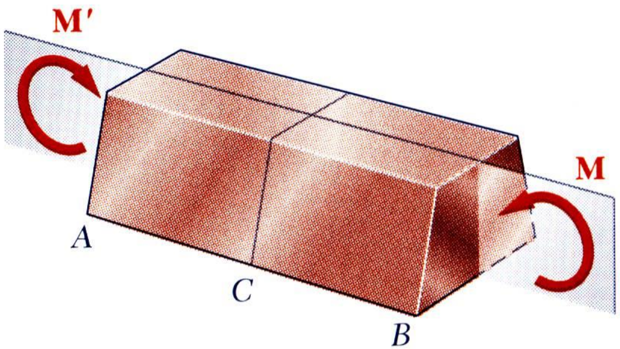


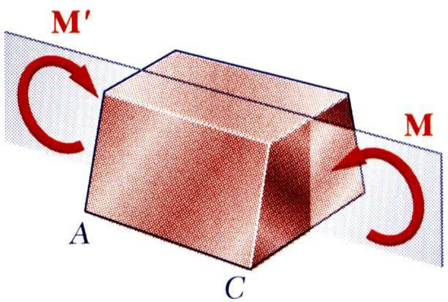

For example, the region between points *C* and *D* of the beam has a constant bending moment M

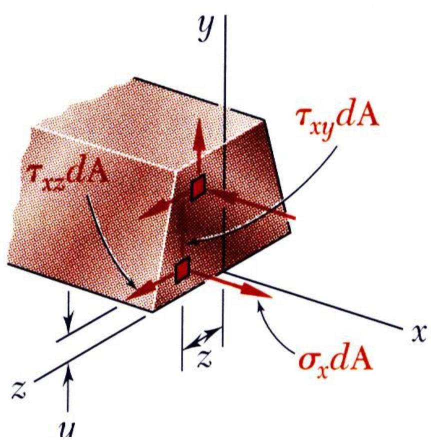



Pure bending occurs only in regions where the transverse shear force *V* is equal to zero

# Symmetric Member in Pure Bending



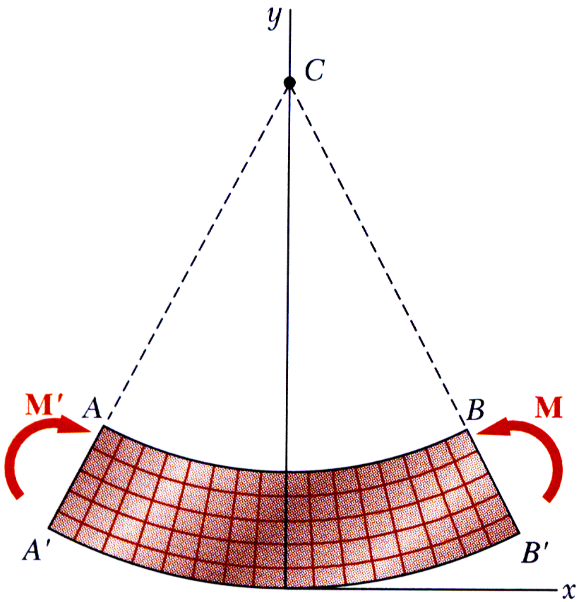

- Internal forces in any cross section are equivalent to a couple. The moment of the couple is the section *bending moment*.
- From statics, a couple M consists of two equal and opposite forces.
- The sum of the components of the forces in any direction is zero.
- The moment is the same about any axis perpendicular to the plane of the couple and zero about any axis contained in the plane.
- These requirements may be applied to the sums of the components and moments of the statically indeterminate elementary internal forces.

$$F_x = \int \sigma_x \, dA = 0$$

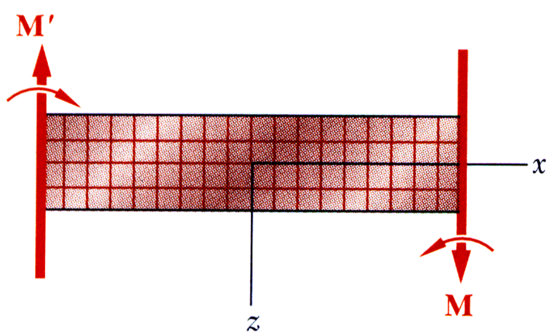
$$M_y = \int z\sigma_x \, dA = 0$$

$$M_z = \int -y\sigma_x \, dA = M$$

# Bending Deformations



(*a*) Longitudinal, vertical section
(plane of symmetry)

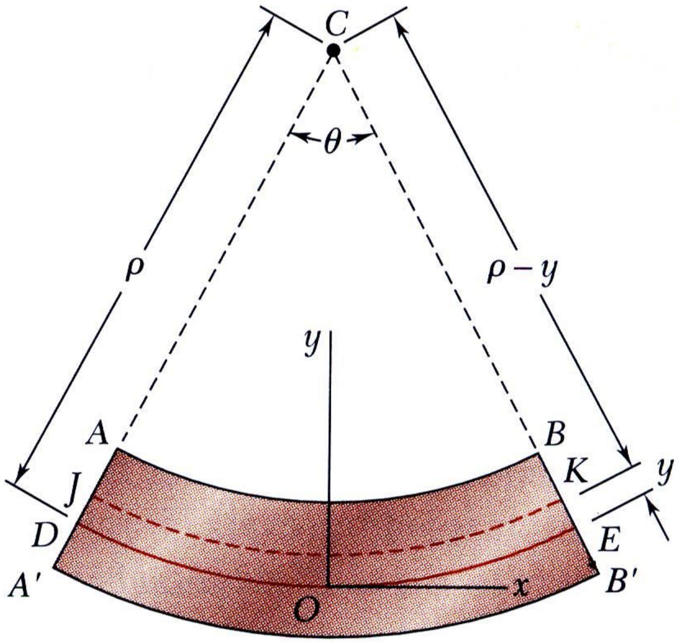


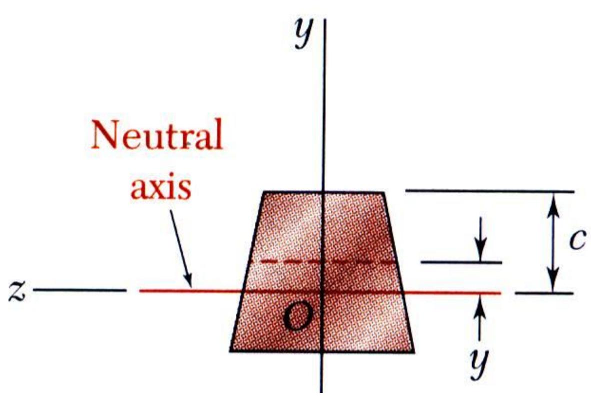
(*b*) Longitudinal, horizontal section

Beam with a plane of symmetry in pure bending:

- member remains symmetric
- bends uniformly to form a circular arc
- cross-sectional plane passes through arc center and remains planar
- length of top decreases and length of bottom increases
- a *neutral surface* must exist that is parallel to the upper and lower surfaces and for which the length does not change
- stresses and strains are negative (compressive) above the neutral plane and positive (tension) below it

# Strain Due to Bending




Consider a beam segment of length *L*.

After deformation, the length of the neutral surface remains *L*. At other sections,

observing that the length of *DE* is equal to the length *L* of the undeformed member, we write

$$L = \rho\theta \qquad \text{(i)}$$

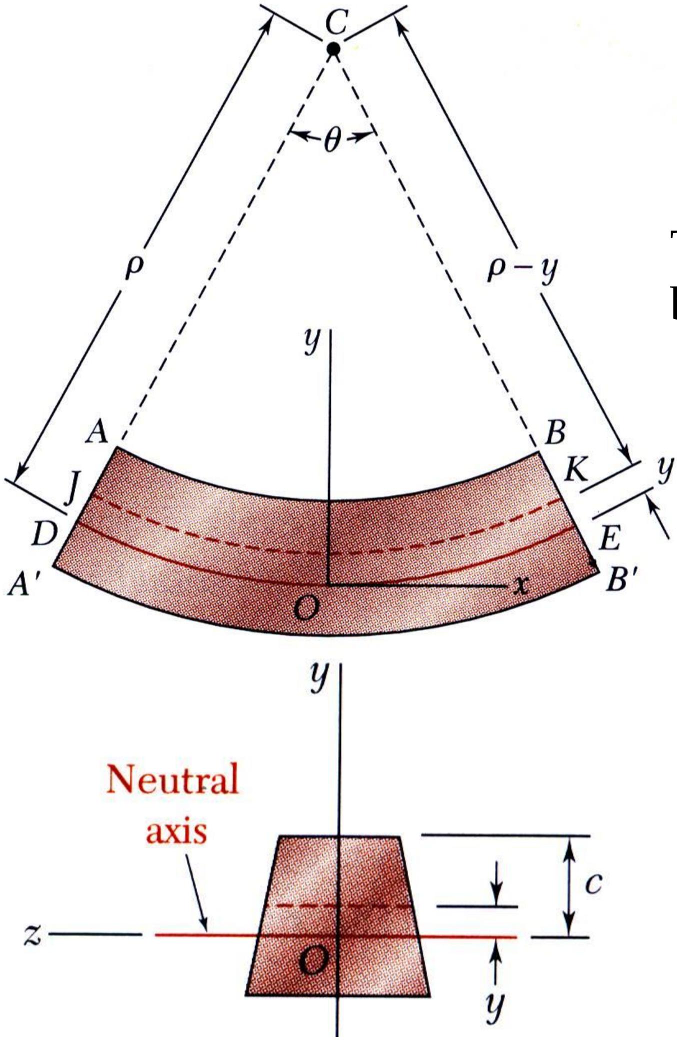
Considering now the arc *JK* located at a distance *y* above the neutral surface, we note that its length *L*' is

$$L' = (\rho - y)\theta \qquad \text{(ii)}$$

Since the original length of arc *JK* was equal to *L*, the deformation of *JK* is

$$\delta = L - L' \qquad \text{(iii)}$$

# Strain Due to Bending




substituting from (i) and (ii) into (iii),

$$\partial = (\rho - y)\theta - \rho\theta = -y\theta$$

The longitudinal strain $\varepsilon_x$ in the elements of $JK$ is obtained by dividing $\partial$ by the original length $L$ of $JK$.

$$\varepsilon_x = \frac{\delta}{L} = -\frac{y\theta}{\rho\theta}$$

$$\varepsilon_x = -\frac{y}{\rho} \qquad \text{(strain varies linearly)}$$

$$\varepsilon_m = \frac{c}{\rho} \quad \text{or} \quad \rho = \frac{c}{\varepsilon_m}$$

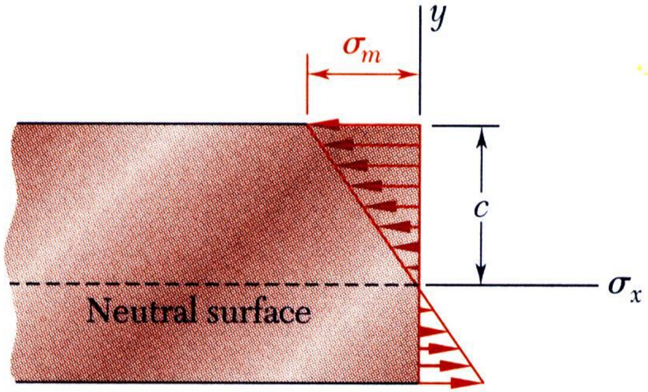
where $\varepsilon_m$ is the *maximum absolute value*

$$Thus; \quad \varepsilon_x = -\frac{y}{c}\varepsilon_m$$

Maximum strain occurs at the outermost fibre located at distance $y = c$ from the Neutral axis

# Stress Due to Bending

- For a linearly elastic material,




$$\sigma_x = E\varepsilon_x = -\frac{y}{c}E\varepsilon_m$$

$$\sigma_x = -\frac{y}{c}\sigma_m \text{ (stress varies linearly)}$$

$\sigma_m$ denotes the *maximum absolute value*

# Stress Due to Bending

- For static equilibrium,

$$dF = \sigma \, dA$$

Neutral axis can be located on the x -section by satisfying the condition that the *resultant force* produced by the stress distribution over the x-sectional area must be equal to *zero*

$$F_x = 0 = \int \sigma_x \, dA = \int -\frac{y}{c}\sigma_m \, dA$$

$$0 = -\frac{\sigma_m}{c}\int y \, dA$$

First moment with respect to neutral plane is zero. Therefore, the neutral surface must pass through the section centroid.

- For static equilibrium,

$$M = \int -y\sigma_x \, dA = \int -y\left(-\frac{y}{c}\sigma_m\right) \, dA$$

$$M = \frac{\sigma_m}{c}\int y^2 \, dA = \frac{\sigma_m I}{c}$$

$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

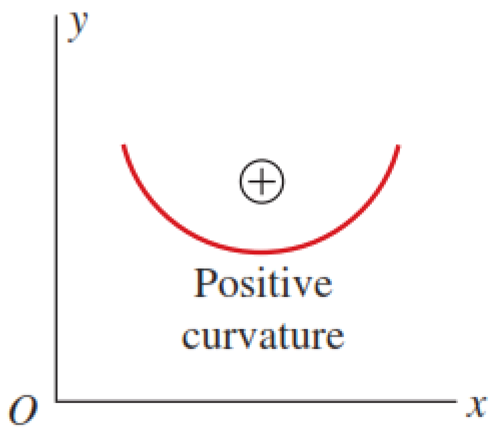
Substituting $\sigma_x = -\frac{y}{c}\sigma_m$

$$\sigma_x = -\frac{My}{I}$$

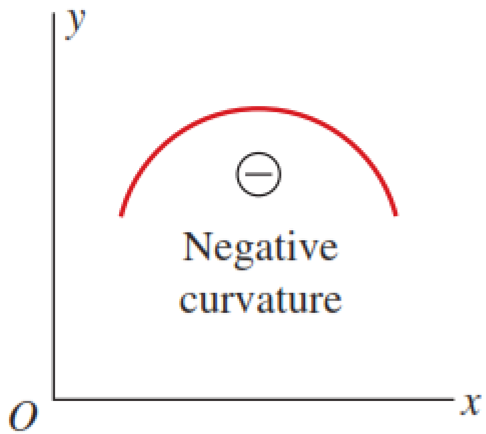
This called the elastic flexure formula

# Sign Convention

The **sign convention for curvature depends upon the orientation of the** coordinate axes. If the *x axis is positive to the right and the y axis is positive* upward, as shown below: then the curvature is positive when the beam is bent concave upward and the center of curvature is above the beam. The converse is true for negative curvature



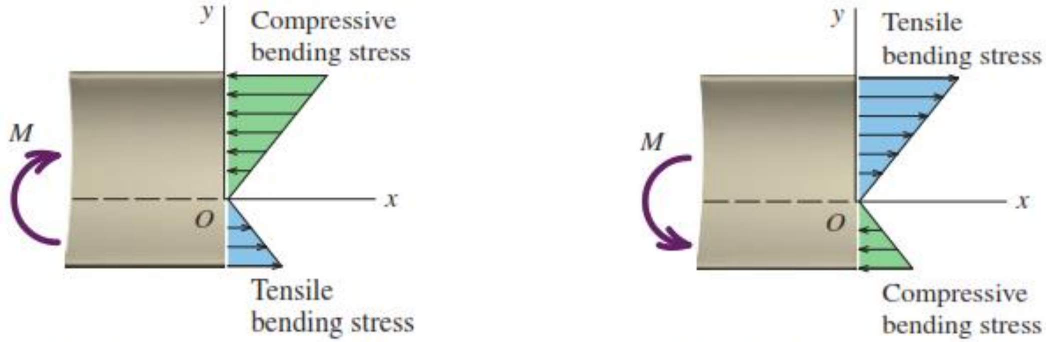

(a)




(b)

# Sign Convention

Relationship between bending moment *M* and bending stress




(*a*) Bending stresses caused by positive *M*

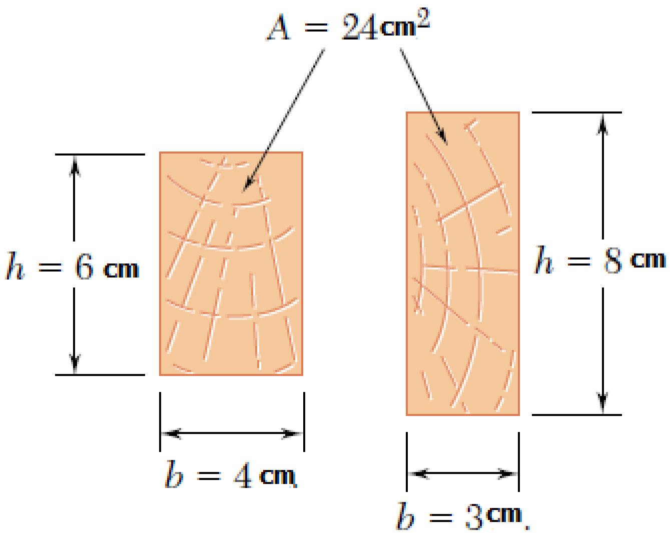
(*b*) Bending stresses caused by negative *M*

Enhanced bending moment sign convention. (a) is a sagging moment and (b) is a hogging moment


(*a*) Positive *M*

(*b*) Negative *M*

# Beam Section Properties




$I = 72 cm^4$
$S = 24 cm^3$

$I = 128 cm^4$
$S = 32\ cm^3$

- The maximum normal stress due to bending,

$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

$I =$ section moment of inertia

The ratio $I/c$ depends only upon the geometry of the cross section.

This ratio is called the *elastic section modulus* and is denoted by *S*.

$$S = \frac{I}{c} = \text{section modulus}$$

Thus; $\sigma_m = \dfrac{M}{S}$

A beam section with a larger section modulus will have a lower maximum stress

# Beam Section Properties

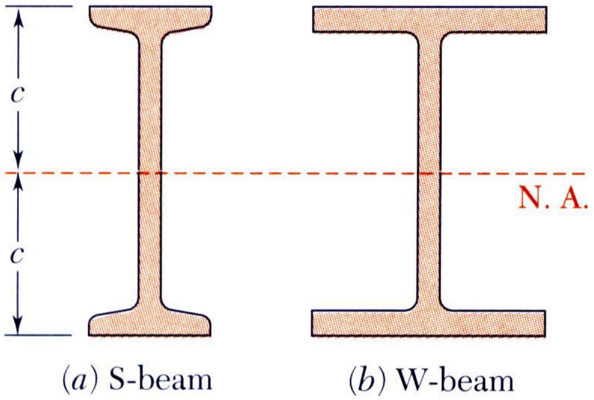
Consider a rectangular beam cross section,

$$S = \frac{I}{c} = \frac{\frac{1}{12}bh^3}{\frac{h}{2}} = \frac{1}{6}bh^2 = \frac{1}{6}Ah$$

Between two beams with the same cross sectional area, the beam with the greater depth will be more effective in resisting bending.

This shows that, of two beams with the same cross-sectional area *A*, the beam with the larger depth *h* will have the larger section modulus and, thus, will be the more effective in resisting bending.

# Beam Section Properties




- Structural steel beams are designed to have a large section modulus.

- In structural steel, standard beams (S-beams) and wide-flange beams (W-beams) are preferred to other shapes because a large portion of their cross section is located far from the neutral axis

- Thus, for a given cross-sectional area and a given depth, their design provides large values of I and, consequently, of S

- Values of the elastic section modulus of commonly manufactured beams can be obtained from tables listing the various geometric properties of such beams

# Beam Section Properties

- To determine the maximum stress $\sigma_m$ in a given section of a standard beam, the engineer needs only to read the value of the elastic section modulus $S$ in a table, and divide the bending moment $M$ in the section by $S$.

- The deformation of the member caused by the bending moment $M$ is measured by the *curvature* of the neutral surface denoted as $\frac{1}{\rho}$. Its reciprocal, $\rho$, is Radius of curvature

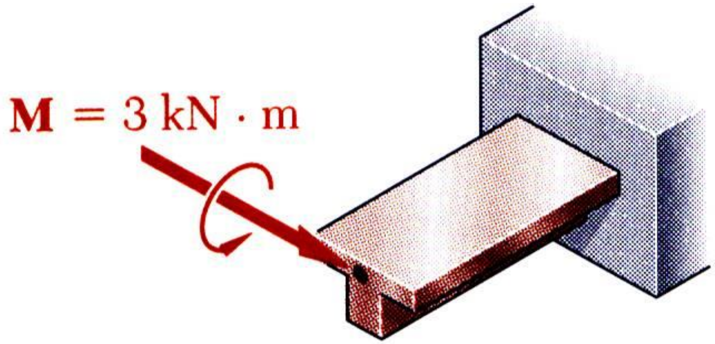
- The curvature is defined as the reciprocal of the radius of curvature $\rho$, and can be obtained by solving

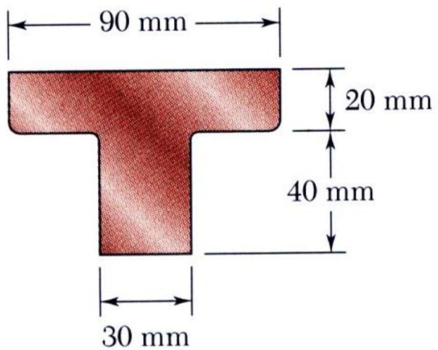
$$\frac{1}{\rho} = \frac{\varepsilon_m}{c}$$

But, in the elastic range, $\varepsilon_m = \sigma_m / E$.

$$\frac{1}{\rho} = \frac{\sigma_m}{Ec} = \frac{1}{Ec}\frac{Mc}{I} \longrightarrow \boxed{\frac{1}{\rho} = \frac{M}{EI}}$$

# Example 1




A cast-iron machine part is acted upon by a 3 kN-m couple. Knowing $E$ = 165 GPa and neglecting the effects of fillets, determine (a) the maximum tensile and compressive stresses, (b) the radius of curvature.

SOLUTION:

- Based on the cross section geometry, calculate the location of the section centroid and moment of inertia.

$$\bar{Y} = \frac{\sum \bar{y}A}{\sum A} \qquad I_{x'} = \sum\left(\bar{I} + Ad^2\right)$$

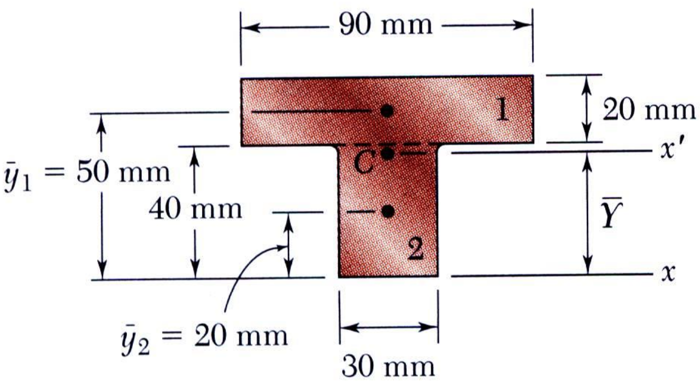
- Apply the elastic flexural formula to find the maximum tensile and compressive stresses.

$$\boldsymbol{\sigma_m = \frac{Mc}{I}}$$

- Calculate the curvature

$$\frac{1}{\rho} = \frac{M}{EI}$$

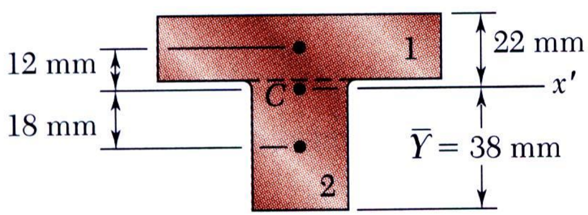
# Example 1 – Cont'd




SOLUTION:

Based on the cross section geometry, calculate the location of the section centroid and moment of inertia.

| | Area, $mm^2$ | $\bar{y}$, mm | $\bar{y}A$, $mm^3$ |
|---|---|---|---|
| 1 | $20 \times 90 = 1800$ | 50 | $90 \times 10^3$ |
| 2 | $40 \times 30 = 1200$ | 20 | $24 \times 10^3$ |
| | $\sum A = 3000$ | | $\sum \bar{y}A = 114 \times 10^3$ |



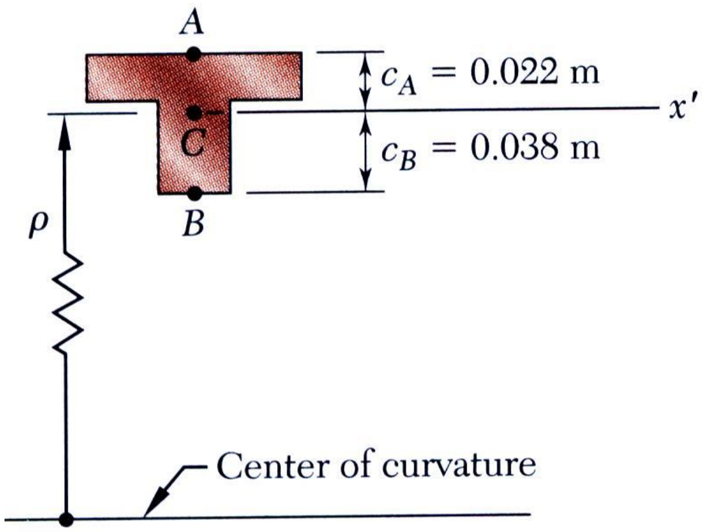

$$\bar{Y} = \frac{\sum \bar{y}A}{\sum A} = \frac{114 \times 10^3}{3000} = 38\,\text{mm}$$

$$I_{x'} = \sum(\bar{I} + Ad^2) = \sum\left(\frac{1}{12}bh^3 + Ad^2\right)$$

$$= \left(\frac{1}{12}90 \times 20^3 + 1800 \times 12^2\right) + \left(\frac{1}{12}30 \times 40^3 + 1200\right.$$

$$\boldsymbol{I = 868 \times 10^3 \text{mm}^4 \;=\; 868 \times 10^{-9} \text{m}^4}$$

# Example 1




- Apply the elastic flexural formula to find the maximum tensile and compressive stresses.

$$\sigma_m = \frac{Mc}{I}$$

$$\sigma_A = \frac{Mc_A}{I} = \frac{3\ \text{kN}\cdot\text{m} \times 0.022\text{m}}{868 \times 10^{-9}\text{m}^4}$$

$$\boxed{\sigma_A = +76.0\ \text{MPa}}$$

$$\sigma_B = -\frac{Mc_B}{I}$$

$$= -\frac{3\ \text{kN}\cdot\text{m} \times 0.038\text{m}}{868 \times 10^{-9}\text{m}^4}$$

$$\boxed{\sigma_B = -131.3\ \text{MPa}}$$

- Calculate the curvature

$$\frac{1}{\rho} = \frac{M}{EI} = \frac{3\ \text{kN}\cdot\text{m}}{(165\ \text{GPa})\left(868 \times 10^{-9}\text{m}^4\right)}$$

$$\boldsymbol{\frac{1}{\rho} = 20.95 \times 10^{-3}\mathbf{m}^{-1}}$$

$$\boldsymbol{\rho = 47.7\ \mathbf{m}}$$