SCHOOL OF ENGINEERING

DEPARTMENT OF CIVIL AND ENVIRONMENTAL ENGINEERING

CEE 3211– MECHANICS OF MATERIALS

DEFLECTION OF BEAMS – Double Integration Method

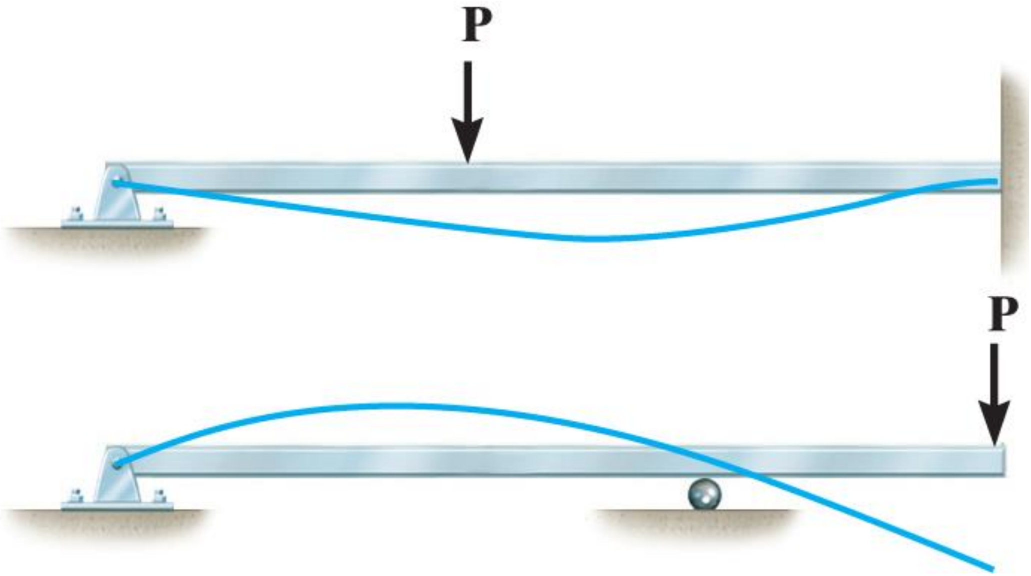
DEFLECTION OF BEAMS

- ✓ Determine the deflection and slope at specific points on beams and shafts, using various analytical methods including:
  - ➢ The integration method

APPLICATIONS

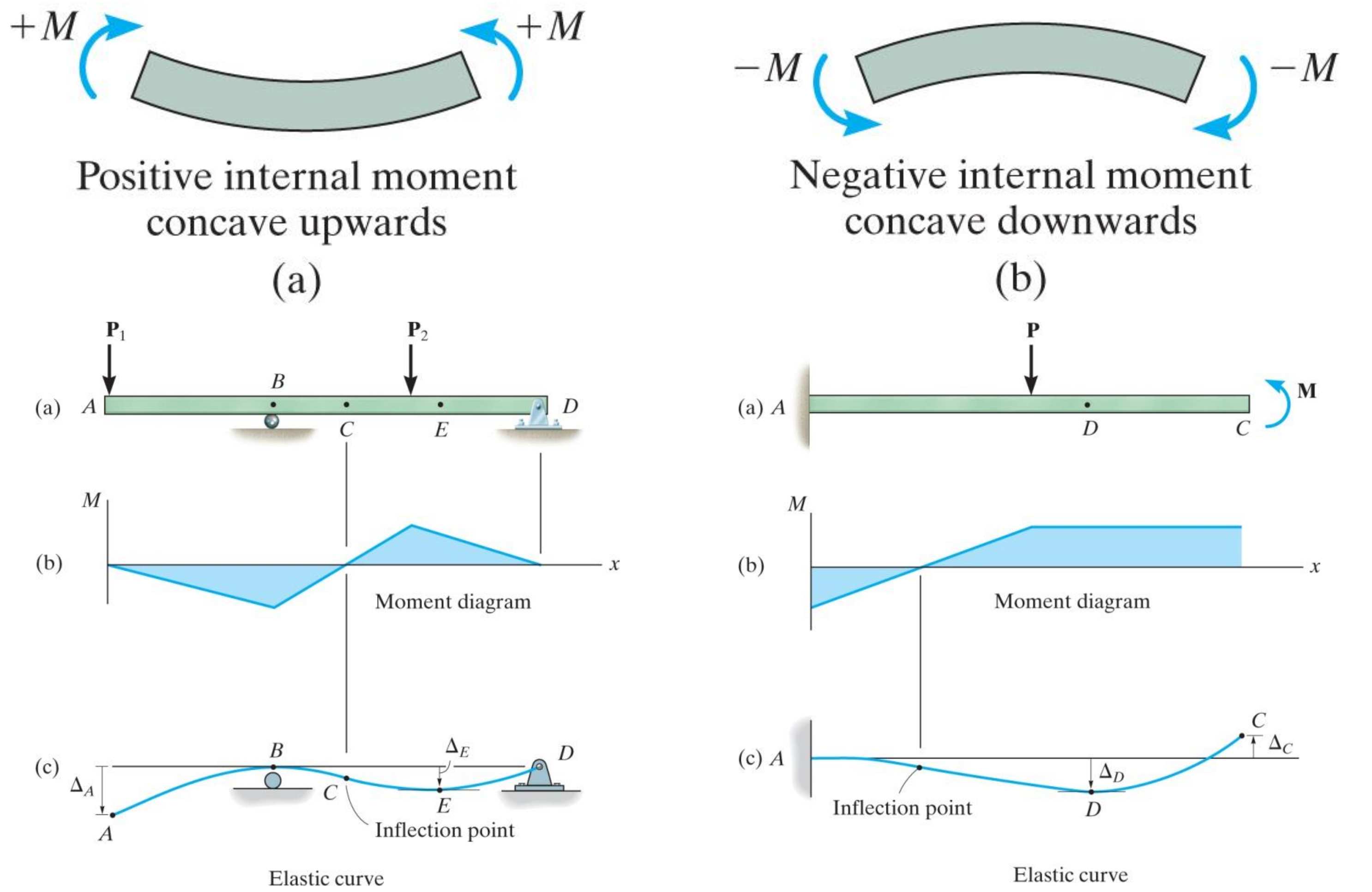
ELASTIC CURVE

- The deflection diagram of the longitudinal axis that passes through the centroid of each cross-sectional area of the beam is called the elastic curve, which is characterized by the deflection and slope along the curve



ELASTIC CURVE (cont)

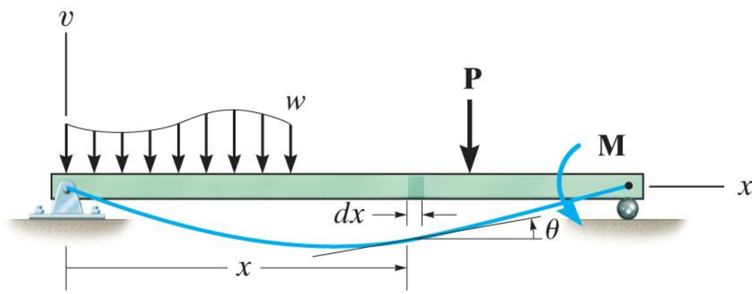
- Moment-curvature relationship:
  - Sign convention:



ELASTIC CURVE (cont)

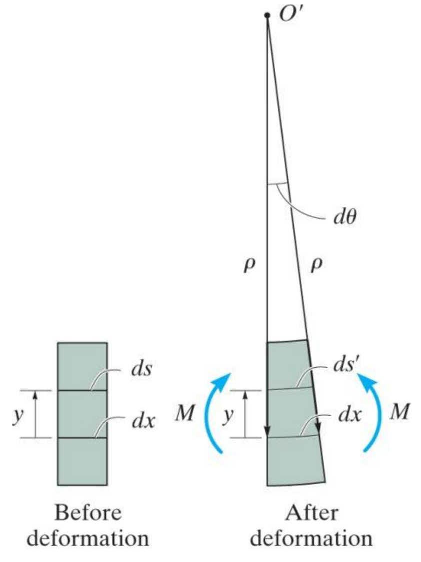
- Consider a segment of width dx, the strain in fibre are ds, located at a position y from the neutral axis is ε = (ds' – ds)/ds. However, ds = dx = ρdθ and ds' = (ρ-y) dθ, and so ε = [(ρ – y) dθ – ρdθ] / (ρdθ), or

$$\frac{1}{\rho} = -\frac{\varepsilon}{y}$$



- Comparing with the Hooke's Law ε = σ / E and the flexure formula σ = -My/I

$$\frac{1}{\rho} = \frac{M}{EI} \quad \text{or} \quad \frac{1}{\rho} = -\frac{\sigma}{Ey}$$



(b)

SLOPE AND DISPLACEMENT BY INTEGRATION

- The equation of the elastic curve is defined by the coordinates v and x. To compute the deflection $v = f(x)$, we must be able to represent the curvature $(1/\rho)$ in terms of v and x.

- Kinematic relationship between radius of curvature ρ and location x:

$$\frac{1}{\rho} = -\frac{d^2v/dx^2}{[1 + (dv/dx)^2]^{3/2}}$$

- Then using the moment curvature equation, we have

$$\frac{M}{EI} = \frac{1}{\rho} = \frac{d^2v/dx^2}{\left[1 + (dv/dx)^2\right]^{3/2}} \approx \frac{d^2v}{dx^2}$$

SLOPE AND DISPLACEMENT BY INTEGRATION

$$\frac{M}{EI} = \frac{d^2v}{dx^2}$$

- The equation can also be written in two alternative forms

- Differentiate each side with respect to x and substitute $V = dM/dx$

$$\frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right) = V(x)$$

- Differentiating again, using $w = dV/dx$

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = w(x)$$

SLOPE AND DISPLACEMENT BY INTEGRATION

- Flexural rigidity (EI) is constant along beam, thus

$$EI\frac{d^4v}{dx^4} = w(x) \text{ [LOAD]}$$

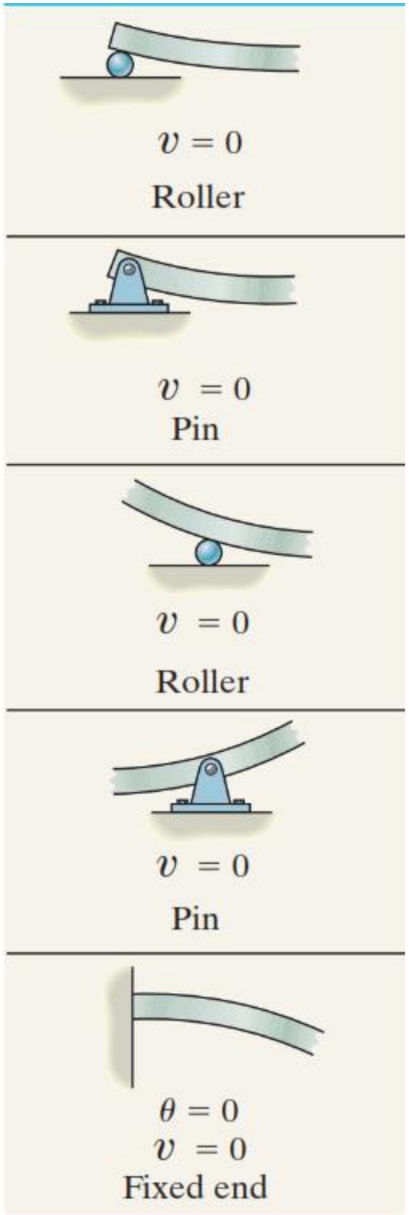
$$EI\frac{d^3v}{dx^3} = V(x) \text{ [Shear]}$$

$$EI\frac{d^2v}{dx^2} = M(x) \text{ [Moment]}$$

- Solution of any of these equations requires successive integrations to obtain v.

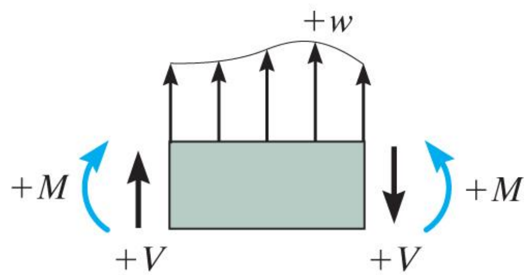
SLOPE AND DISPLACEMENT BY INTEGRATION (cont)

- Boundary Conditions:
  - The integration constants can be determined by imposing the boundary conditions, or
  - Continuity condition at specific locations
- Note, if the beam is supported by a *roller* or *pin*, then it is required that the displacement be *zero* at these points.
- At the fixed support, the *slope* and *displacement* are both *zero*.



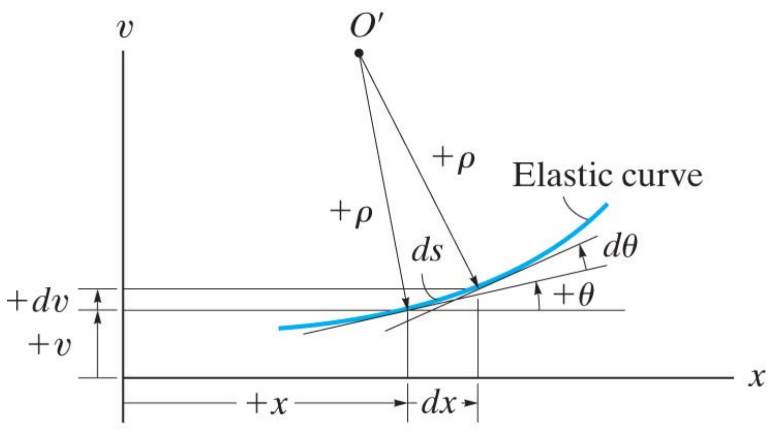
SLOPE AND DISPLACEMENT BY INTEGRATION (cont)

- Sign convention:



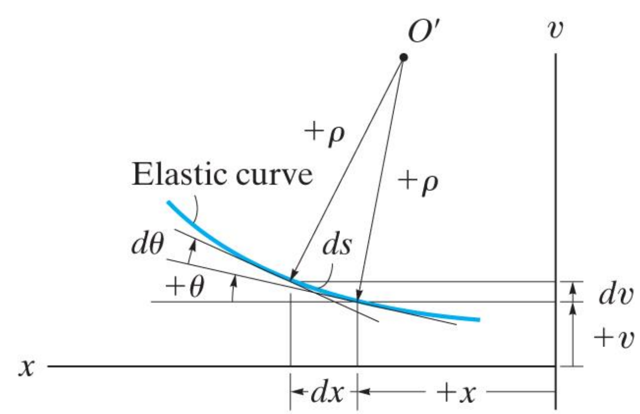
Positive sign convention

(a)



Positive sign convention

(b)



Positive sign convention

(c)

SUMMARY

$$\text{Deflection} = v$$

$$\text{Slope} = \frac{dv}{dx} = \theta$$

$$\text{Moment } M = EI\frac{d^2v}{dx^2}$$

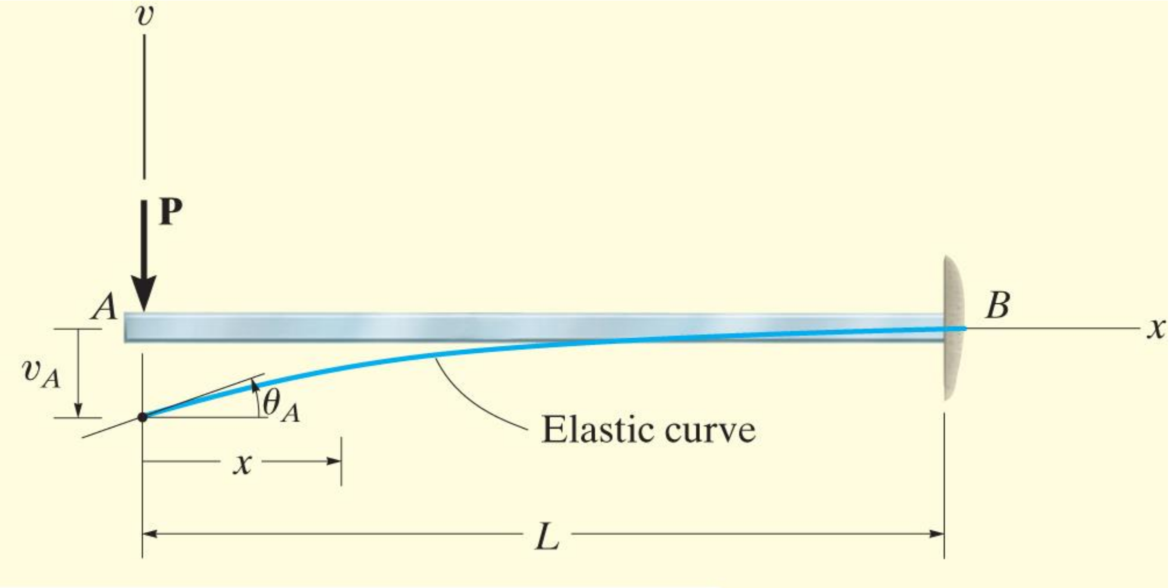
$$\text{Shear } V = \frac{dM}{dx} = EI\frac{d^3v}{dx^3} \quad (\text{for } EI \text{ constant})$$

$$\text{Load } w = \frac{dV}{dx} = EI\frac{d^4v}{dx^4} \quad (\text{for } EI \text{ constant})$$

EXAMPLE 1

The cantilevered beam shown below is subjected to a vertical load **P** at its end.

1. Determine the equation of the elastic curve. EI is constant.
2. Determine max slope and deflection given the following data: $L = 5$ m; Load; $P = 30$ kN) and designed without a factor of safety by assuming the allowable normal stress is equal to the yield stress is 250 MPa; then a W310 x 39 would be found to be adequate ($I\ =\ 84.4(10^6)\text{mm}^4$)



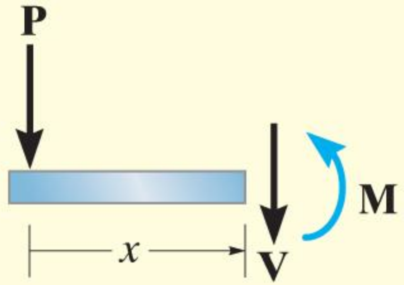
EXAMPLE 1 (cont)

Solutions

- Elastic Curve: shown in the Question figure

- Moment Function: From the free-body diagram, with **M** acting in the *positive direction*, we have

$$M = -Px$$



(b)

- Slope and Elastic Curve

- Applying $EI\frac{d^2v}{dx^2} = M(x)$ and integrating twice yields

$$EI\frac{d^2v}{dx^2} = -Px \quad (1)$$

$$EI\frac{dv}{dx} = -\frac{Px^2}{2} + C_1 \quad (2)$$

$$EIv = -\frac{Px^3}{6} + C_1x + C_2 \quad (3)$$

EXAMPLE 1 (cont)

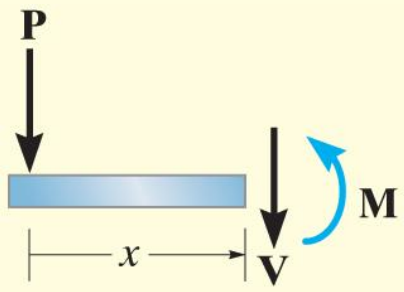
Solutions

- Using the boundary conditions $dv/dx = 0$ at $x = L$ and $v = 0$ at $x = L$, equations 2 and 3 become

$$0 = -\frac{PL^2}{2} + C_1$$

$$0 = -\frac{PL^3}{6} + C_1 L + C_2$$

$$\Rightarrow C_1 = \frac{PL^2}{2} \text{ and } C_2 = -\frac{PL^3}{3}$$



(b)

- Substituting these results, with $\theta = dv/dx$, we get

$$\theta = \frac{P}{2EI}\left(L^2 - x^2\right)$$

Elastic Curve Eqn: $$v = \frac{P}{6EI}\left(-x^3 + 3L^2 x - 2L^3\right) \quad \text{(Ans)}$$

EXAMPLE 1 (cont)

Solutions

- Maximum slope and displacement occur at for which $A(x = 0)$,

$$\theta_A = \frac{PL^2}{2EI} \quad (4)$$

$$v_A = -\frac{PL^3}{3EI} \quad (5)$$

- Substituting the values of ($L = 5$ m; Load; $P = 30$ kN, $I = 84.4(10^6)\text{mm}^4$)

$$\theta_A = \frac{30{,}000(5{,}000)^2}{2[200{,}000][84.4(10^6)]} = \mathbf{0.0222\ rad}$$

$$v_A = -\frac{30{,}000(5{,}000)^3}{3[200{,}000][84.4(10^6)]} = \mathbf{-74.1\ mm}$$

EXAMPLE 3

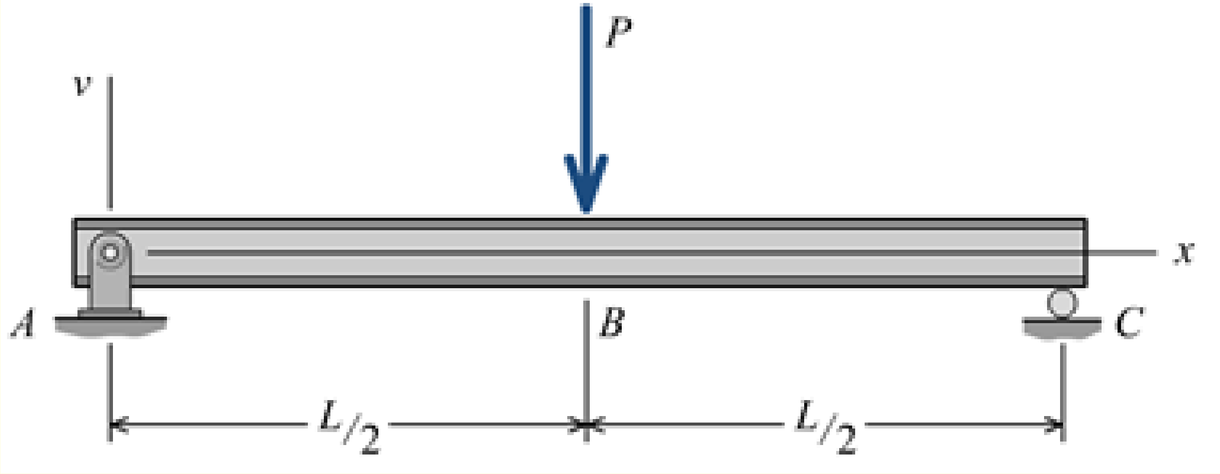
For the beam and loading shown below, use the double-integration method to determine

(a) the equation of the elastic curve for segment AB of the beam.

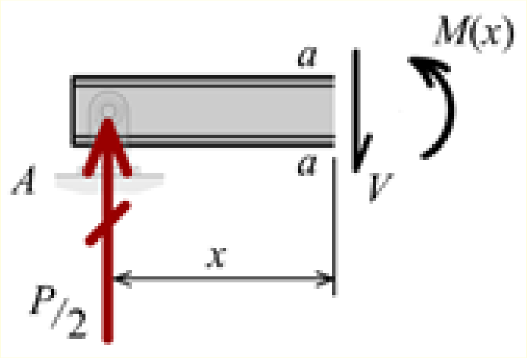
(b) the deflection at B.

(c) the slope at A.

Assume that EI is constant for the beam.. *EI* is constant.



EXAMPLE 2



Integration of moment equation:

$$EI\frac{d^2v}{dx^2} = M(x) = \frac{P}{2}x$$

$$EI\frac{dv}{dx} = \frac{Px^2}{4} + C_1 \quad \longrightarrow \quad \text{(a)}$$

$$EI\,v = \frac{Px^3}{12} + C_1x + C_2 \quad \longrightarrow \quad \text{(b)}$$

BOUNDARY CONDITIONS

$$v = 0 \quad \text{at} \quad x = 0$$

$$\frac{dv}{dx} = 0 \quad \text{at} \quad x = \frac{L}{2}$$

EXAMPLE 2

Substitute $x = L/2$ and $dv/dx = 0$ into Eq. (a) to determine C_1:

$$EI(0) = \frac{P(L/2)^2}{4} + C_1 \qquad \therefore C_1 = -\frac{PL^2}{16}$$

Substitute $x = 0$ and $v = 0$ into Eq. (b) to determine C_2:

$$EI(0) = \frac{P(0)^3}{12} - \frac{PL^2(0)}{16} + C_2 \qquad \therefore C_2 = 0$$

EXAMPLE 2

(a) – **ELASTIC CURVE**

$$EI\,v = \frac{Px^3}{12} - \frac{PL^2x}{16} \qquad \therefore v = \boxed{-\frac{Px}{48EI}\left[3L^2 - 4x^2\right]} \qquad (0 \le x \le \frac{L}{2})$$ **Ans.**

(b) – **Deflection @ b**

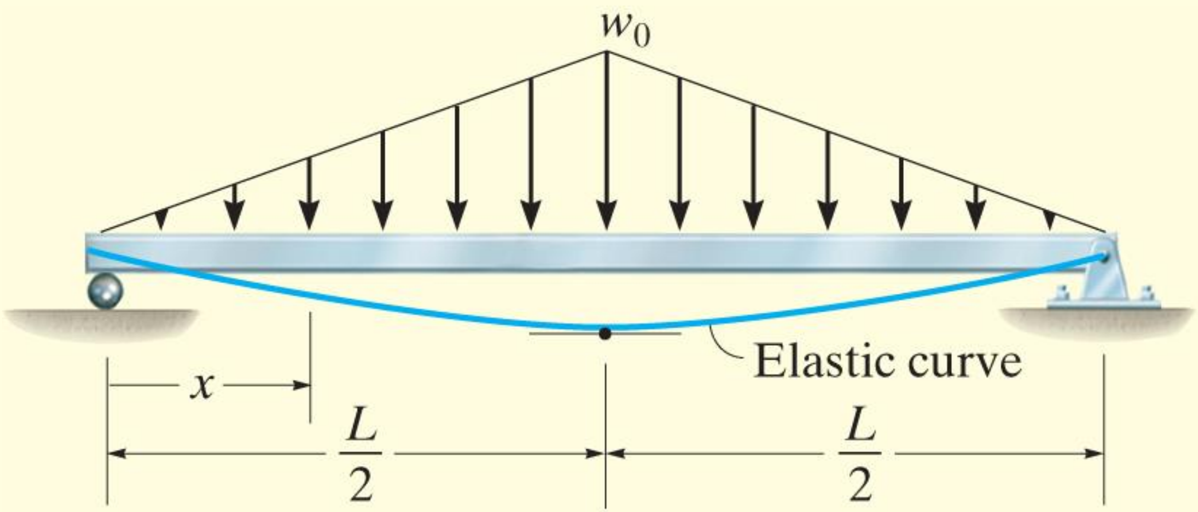
$$v_B = -\frac{P(L/2)}{48EI}\left[3L^2 - 4\left(\frac{L}{2}\right)^2\right] = \boxed{-\frac{PL^3}{48EI}}$$ **Ans**

(b) – **Slope @ A**

$$\left.\frac{dv}{dx}\right|_A = \theta_A = \frac{P(0)^2}{4EI} - \frac{PL^2}{16EI} = \boxed{-\frac{PL^2}{16EI}}$$ **Ans.**

EXAMPLE 2

The simply supported beam shown below supports the triangular distributed loading. Determine its maximum deflection. *EI* is constant.



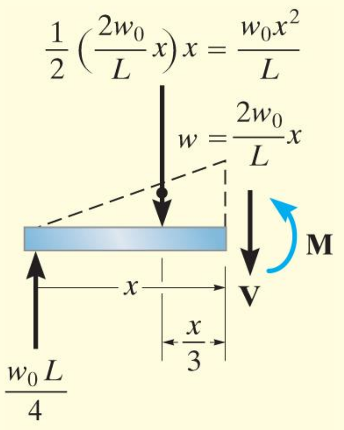
(a)

EXAMPLE 3 (cont)

Solutions

- Due to symmetry only one *x* coordinate is needed for the solution,

$$0 \le x \le L/2$$



(b)

- The equation for the distributed loading is $w = \frac{2w_0}{L}x$.

- Hence

$$+\sum M_{NA} = 0; \quad M + \frac{w_0x^2}{L}\left(\frac{x}{3}\right) - \frac{w_0L}{4}(x) = 0$$

$$M = -\frac{w_0x^2}{3L} + \frac{w_0L}{4}x$$

EXAMPLE 3 (cont)

Solutions

- Integrating twice, we have

$$EI\frac{d^2v}{dx^2}=M=-\frac{w_0}{3L}x^3+\frac{w_0L}{4}x$$

$$EI\frac{dv}{dx}=-\frac{w_0}{12L}x^4+\frac{w_0L}{8}x^2+C_1$$

$$EIv=-\frac{w_0}{60L}x^5+\frac{w_0L}{24}x^3+C_1x+C_2$$

- For boundary condition,

  - $v=0, x=0$ and
  - $dv/dx=0, x=L/2$

$$C_1=-\frac{5w_0L^3}{192}$$

$$C_2=0$$

EXAMPLE 3 (cont)

Solutions

- Hence

$$EIv = -\frac{w_0}{60L}x^5 + \frac{w_0 L}{24}x^3 - \frac{5w_0 L^3}{192}x$$

- For maximum deflection at $x = L/2$,

$$v_{\max} = -\frac{w_0 L^4}{120EI} \quad \text{(Ans)}$$