SCHOOL OF ENGINEERING

DEPARTMENT OF CIVIL AND ENVIRONMENTAL ENGINEERING

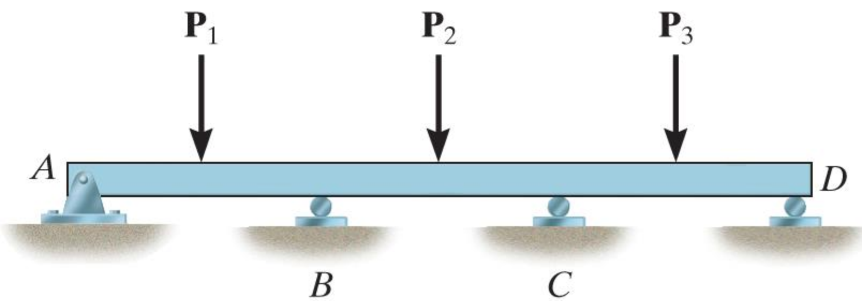
CEE 3211– MECHANICS OF MATERIALS

STATICALLY INDETERMINATE BEAMS

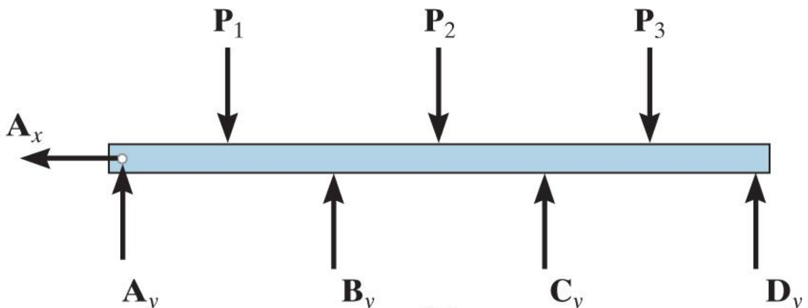
STATISTICALLY INDETERMINATE BEAMS AND SHAFT

- **Definition**:

  A member of any type is classified *statically indeterminate* if the number of unknown reactions **exceeds** the available number of equilibrium equations, e.g. a continuous beam having 4 supports



(a)



(b)

STATISTICALLY INDETERMINATE BEAMS (cont)

Strategy:

- The additional support reactions on the beam or shaft that are not needed to keep it in stable equilibrium are called *redundants*. It is first necessary to specify those redundant from conditions of geometry known as compatibility conditions.

- Once determined, the redundants are then applied to the beam, and the remaining reactions are determined from the equations of equilibrium.

USE OF THE METHOD OF SUPERPOSITION

Procedures:

Elastic Curve

- Specify the unknown redundant forces or moments that must be removed from the beam in order to make it statistically determinate and stable.

- Using the principle of superposition, draw the statistically indeterminate beam and show it equal to a sequence of corresponding *statistically determinate* beams.

USE OF THE METHOD OF SUPERPOSITION (cont)

Procedures:

Elastic Curve (cont)

- The first of these beams, the primary beam, supports the same external loads as the statistically indeterminate beam, and each of the other beams “added” to the primary beam shows the beam loaded with a separate redundant force or moment.

- Sketch the deflection curve for each beam and indicate symbolically the displacement or slope at the point of each redundant force or moment.

USE OF THE METHOD OF SUPERPOSITION (cont)

Procedures:

Compatibility Equations

- Write a compatibility equation for the displacement or slope at each point where there is a redundant force or moment.
- Determine all the displacements or slopes using an appropriate method (integration, Macaulay, or Moment Area method).

USE OF THE METHOD OF SUPERPOSITION (cont)

Procedures:

Compatibility Equations (cont)

- Substitute the results into the compatibility equations and solve for the unknown redundant.

- If the numerical value for a redundant is *positive*, it has the same *sense of direction* as originally assumed. Similarly, a negative numerical value indicates the redundant acts *opposite* to its assumed *sense of direction.*

USE OF THE METHOD OF SUPERPOSITION (cont)

Procedures:

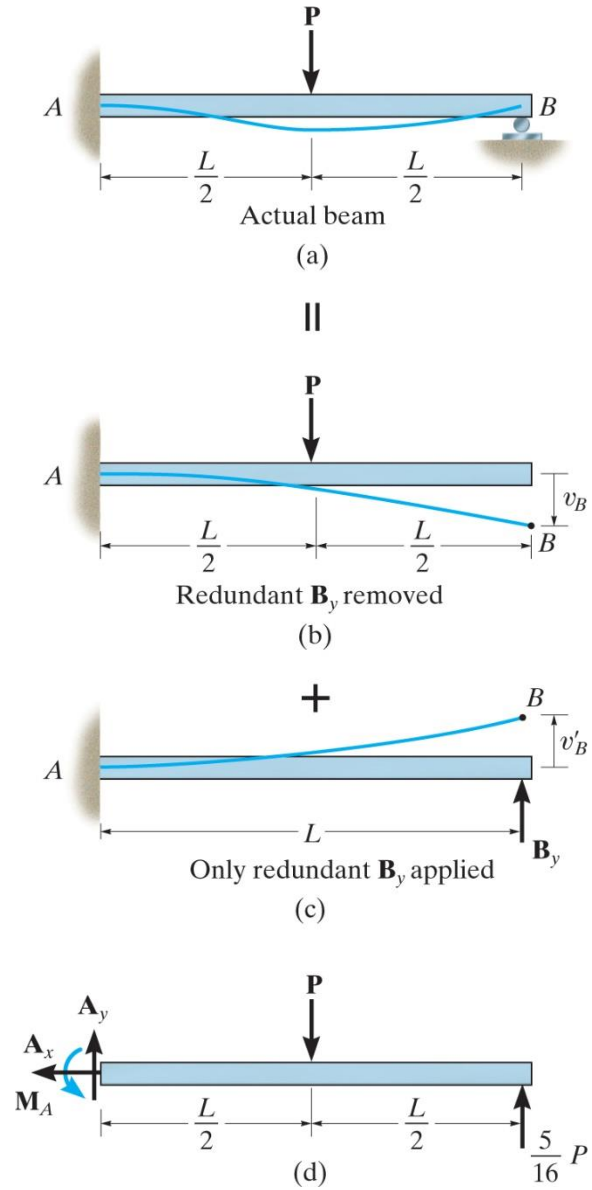
Equilibrium Equations

- Once the redundant forces and/or moments have been determined, the remaining unknown reactions can be found from the equations of equilibrium applied to the loadings shown on the beam's free body diagram.

$$0 = -v_B + v'_B$$

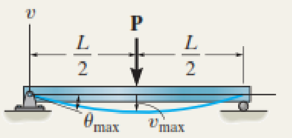
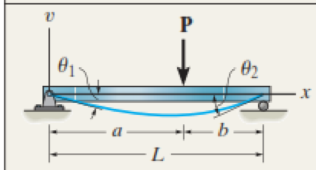
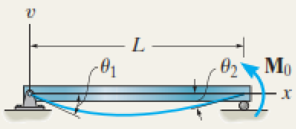
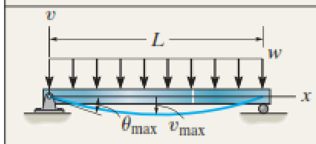
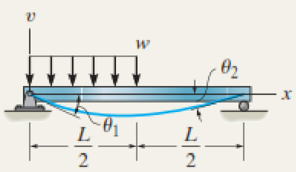
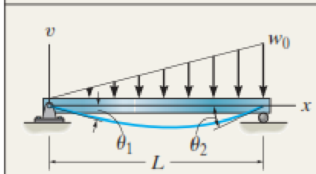
From Literature; $\quad v_B = \dfrac{5PL^3}{48EI} \qquad v'_B = \dfrac{B_y L^3}{3EI}$

$B_y = \dfrac{5}{16}P$; Thus $A_x = 0$; $A_y = \dfrac{11}{16}P$; $M_A = \dfrac{3}{16}PL$



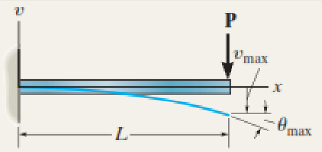
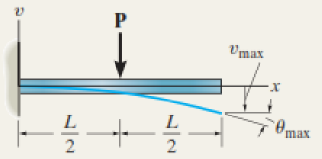
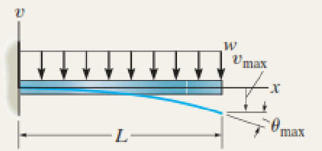
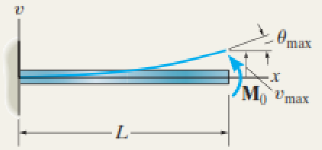
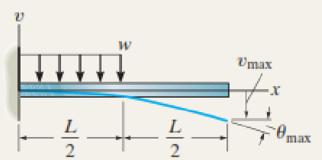
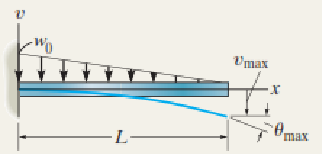
DEFLECTIONS TABLES OF STANDARD BEAMS

Simply Supported Beam Slopes and Deflections

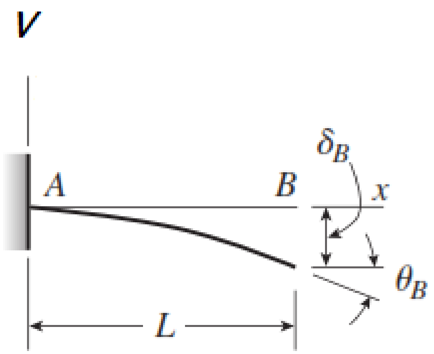
Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \frac{-PL^2}{16EI}$	$v_{max} = \frac{-PL^3}{48EI}$	$v = \frac{-Px}{48EI}(3L^2 - 4x^2)$ $0 \le x \le L/2$
	$\theta_1 = \frac{-Pab(L+b)}{6EIL}$ $\theta_2 = \frac{Pab(L+a)}{6EIL}$	$v\Big\rvert_{x=a} = \frac{-Pba}{6EIL}(L^2 - b^2 - a^2)$	$v = \frac{-Pbx}{6EIL}(L^2 - b^2 - x^2)$ $0 \le x \le a$
	$\theta_1 = \frac{-M_0L}{6EI}$ $\theta_2 = \frac{M_0L}{3EI}$	$v_{max} = \frac{-M_0L^2}{\sqrt{243}EI}$ at $x = 0.5774L$	$v = \frac{-M_0x}{6EIL}(L^2 - x^2)$
	$\theta_{max} = \frac{-wL^3}{24EI}$	$v_{max} = \frac{-5wL^4}{384EI}$	$v = \frac{-wx}{24EI}(x^3 - 2Lx^2 + L^3)$
	$\theta_1 = \frac{-3wL^3}{128EI}$ $\theta_2 = \frac{7wL^3}{384EI}$	$v\Big\rvert_{x=L/2} = \frac{-5wL^4}{768EI}$ $v_{max} = -0.006563\frac{wL^4}{EI}$ at $x = 0.4598L$	$v = \frac{-wx}{384EI}(16x^3 - 24Lx^2 + 9L^3)$ $0 \le x \le L/2$ $v = \frac{-wL}{384EI}(8x^3 - 24Lx^2 + 17L^2x - L^3)$ $L/2 \le x < L$
	$\theta_1 = \frac{-7w_0L^3}{360EI}$ $\theta_2 = \frac{w_0L^3}{45EI}$	$v_{max} = -0.00652\frac{w_0L^4}{EI}$ at $x = 0.5193L$	$v = \frac{-w_0x}{360EIL}(3x^4 - 10L^2x^2 + 7L^4)$

DEFLECTIONS TABLES OF STANDARD BEAMS

Cantilevered Beam Slopes and Deflections

Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \dfrac{-PL^2}{2EI}$	$v_{max} = \dfrac{-PL^3}{3EI}$	$v = \dfrac{-Px^2}{6EI}(3L - x)$
	$\theta_{max} = \dfrac{-PL^2}{8EI}$	$v_{max} = \dfrac{-5PL^3}{48EI}$	$v = \dfrac{-Px^2}{6EI}\left(\frac{3}{2}L - x\right) \quad 0 \le x \le L/2$ $v = \dfrac{-PL^2}{24EI}\left(3x - \frac{1}{2}L\right) \quad L/2 \le x \le L$
	$\theta_{max} = \dfrac{-wL^3}{6EI}$	$v_{max} = \dfrac{-wL^4}{8EI}$	$v = \dfrac{-wx^2}{24EI}(x^2 - 4Lx + 6L^2)$
	$\theta_{max} = \dfrac{M_0L}{EI}$	$v_{max} = \dfrac{M_0L^2}{2EI}$	$v = \dfrac{M_0x^2}{2EI}$
	$\theta_{max} = \dfrac{-wL^3}{48EI}$	$v_{max} = \dfrac{-7wL^4}{384EI}$	$v = \dfrac{-wx^2}{24EI}\left(x^2 - 2Lx + \frac{3}{2}L^2\right)$ $0 \le x \le L/2$ $v = \dfrac{-wL^3}{192EI}(4x - L/2)$ $L/2 \le x \le L$
	$\theta_{max} = \dfrac{-w_0L^3}{24EI}$	$v_{max} = \dfrac{-w_0L^4}{30EI}$	$v = \dfrac{-w_0x^2}{120EIL}(10L^3 - 10L^2x + 5Lx^2 - x^3)$

DEFLECTIONS TABLES OF STANDARD BEAMS



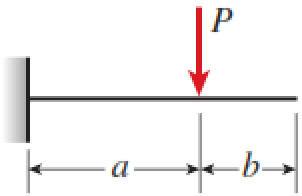
$v =$ deflection in the y direction (positive upward)

$v' = dv/dx =$ slope of the deflection curve

$\delta_B = -v(L) =$ deflection at end B of the beam (positive downward)

$\theta_B = -v'(L) =$ angle of rotation at end B of the beam (positive clockwise)

$EI =$ constant



$$v = -\frac{Px^2}{6EI}(3a - x) \qquad v' = -\frac{Px}{2EI}(2a - x) \qquad (0 \le x \le a)$$

$$v = -\frac{Pa^2}{6EI}(3x - a) \qquad v' = -\frac{Pa^2}{2EI} \qquad (a \le x \le L)$$

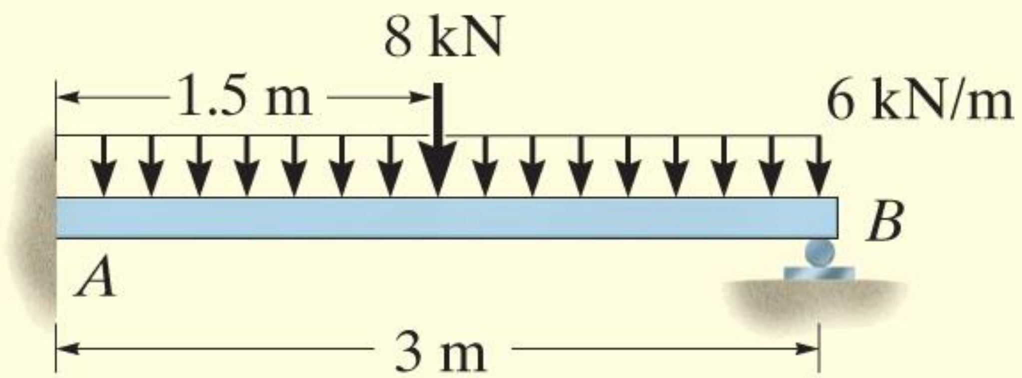
$$\text{At } x = a\text{:} \qquad v = -\frac{Pa^3}{3EI} \qquad v' = -\frac{Pa^2}{2EI}$$

$$\delta_B = \frac{Pa^2}{6EI}(3L - a) \qquad \theta_B = \frac{Pa^2}{2EI}$$

EXAMPLE 2

Determine the reactions at the roller support *B* of the beam shown in Fig. below, then draw the shear and moment diagrams. *EI* is constant.

(a)



Actual beam

EXAMPLE 2 (cont)

Solutions

- By inspection, the beam is statically indeterminate to the first degree.

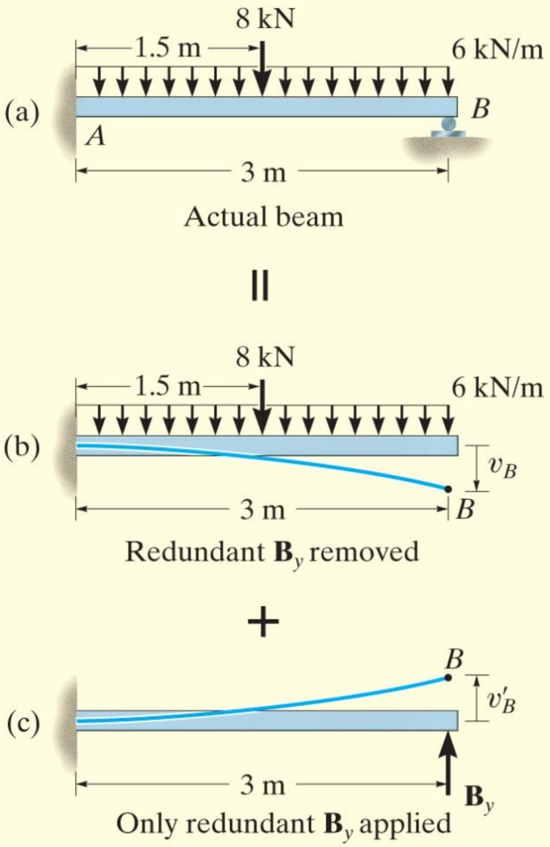
- Taking positive displacement as downward, the compatibility equation at *B* is

$$(+\downarrow) \qquad 0 = v_B - v'_B \qquad (1)$$

- Displacements can be obtained from tables.

$$v_B = \frac{wL^4}{8EI} + \frac{5PL^3}{48EI} = \frac{83.25\ \mathrm{kN\cdot m^3}}{\mathrm{EI}} \downarrow$$

$$v'_B = \frac{PL^3}{3EI} = \frac{(9\ \mathrm{m^3})B_y}{EI} \uparrow$$

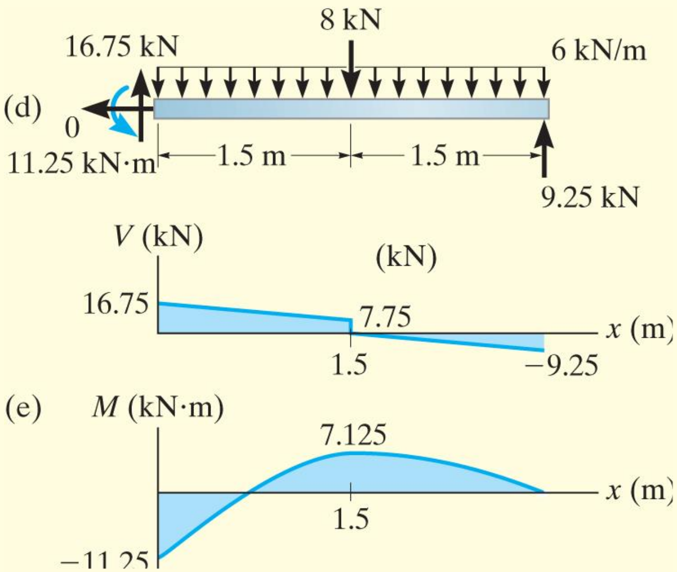


EXAMPLE 2 (cont)

Solutions

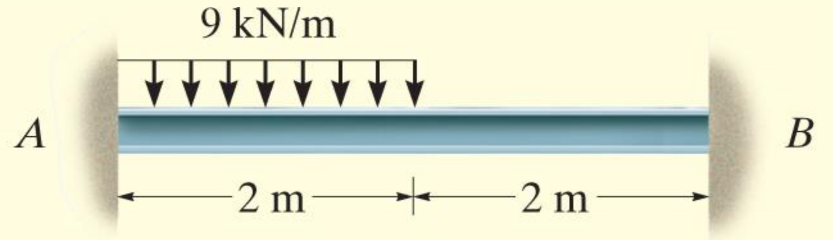
- Substituting into Eq. 1 and solving yields

$$\boxed{\begin{aligned} 0 &= \frac{83.25}{EI} - \frac{9B_y}{EI} \\ B_y &= 9.25\,\text{kN} \end{aligned}}$$



EXAMPLE 3

Determine the moment at *B* for the beam shown in Fig. below. *EI* is constant. Neglect the effects of axial load.

(a) 

Actual beam

EXAMPLE 3 (cont)

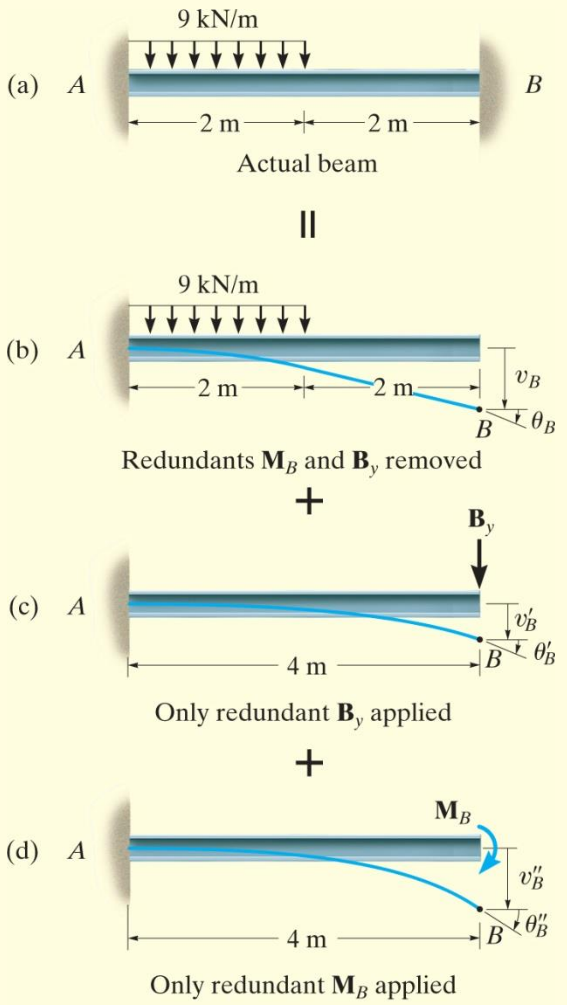
Solutions

- Since the axial load on the beam is neglected, there will be a vertical force and moment at *A* and *B*.

- Referring to the displacement and slope at *B*, we require

$$(+) \qquad 0 = \theta_B + \theta'_B + \theta''_B \qquad (1)$$

$$(+\downarrow) \qquad 0 = v_B + v'_B + v''_B \qquad (2)$$



EXAMPLE 3 (cont)

Solutions

- Use standard beam deflections to calculate slopes and displacements,

$$\theta_B = \frac{wL^3}{48EI} = \frac{12\ \mathrm{kN\cdot m^3}}{\mathrm{EI}}$$

$$v_B = \frac{7wL^4}{384EI} = \frac{42\ \mathrm{kN\cdot m^3}}{EI} \downarrow$$

$$\theta'_B = \frac{PL^2}{2EI} = \frac{8B_y}{EI}$$

$$v'_B = \frac{PL^3}{3EI} = \frac{21.33B_y}{EI} \downarrow$$

$$\theta''_B = \frac{ML}{EI} = \frac{4M_B}{EI}$$

$$v''_B = \frac{ML^2}{2EI} = \frac{8M_B}{EI} \downarrow$$

EXAMPLE 3 (cont)

Solutions

- Substituting these values into Eqs. 1 and 2 and cancelling out the common factor *EI*, we get

$$\circlearrowleft (+) \qquad 0 = 12 + 8B_y + 4M_B$$

$$(+\downarrow) \qquad 0 = 42 + 21.33B_y + 8M_B$$

- Solving these equations simultaneously gives

$$\boxed{\begin{aligned} B_y &= 3.375\,\text{kN} \\ M_B &= 3.75\,\text{kN}\cdot\text{m} \quad (\text{Ans}) \end{aligned}}$$

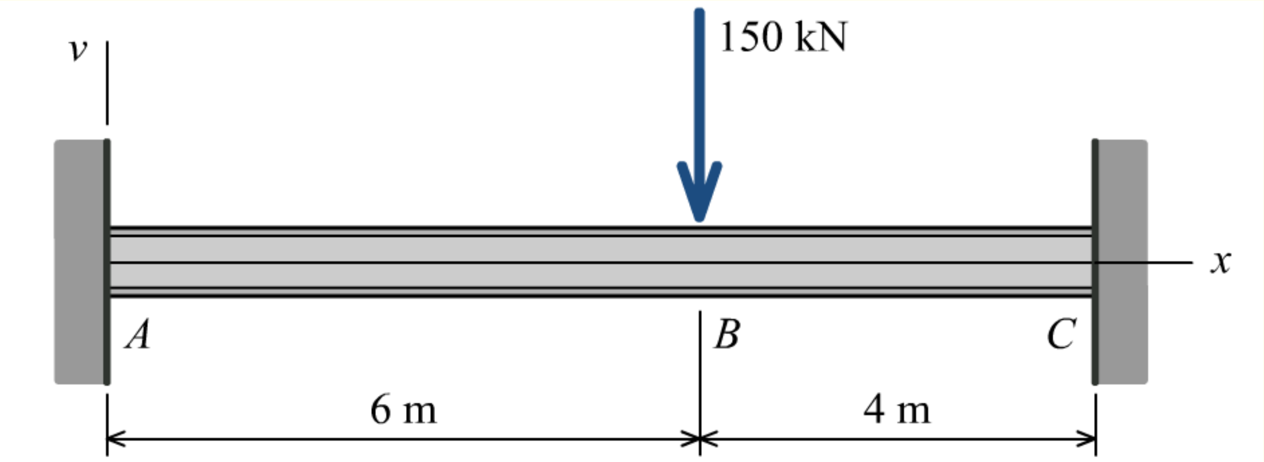
EXAMPLE 4

Solutions

A structural steel beam [$E = 200$ GPa; $I = 554 \times 10^6$ mm^4] is loaded and supported as shown below. Determine:

i. the force and moment reactions at supports A and C.

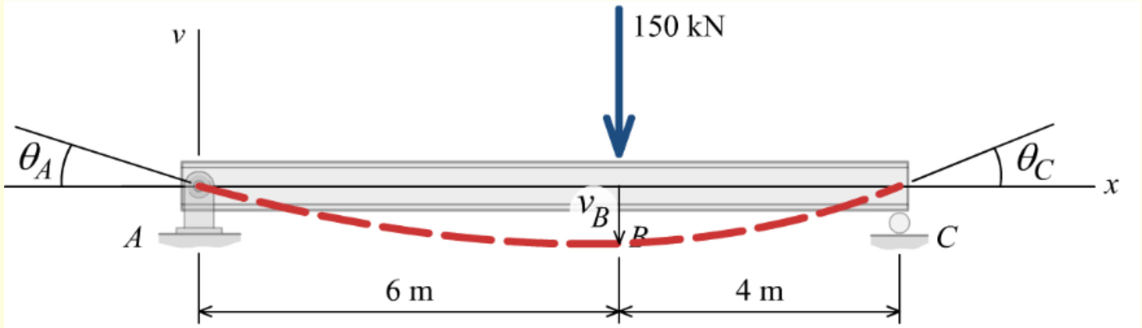
ii. the deflection of the beam at B.



EXAMPLE 4

Solutions

- Reactions at A and C. Choose the moment reactions at A and C as the redundants.
- This will leave a simply supported beam between A and C as the released beam
- Determine the slopes at A and C caused by the 150-kN concentrated load.



$$\theta_A = -\frac{Pb(L^2 - b^2)}{6LEI} \quad \text{and} \quad \theta_C = \frac{Pa(L^2 - a^2)}{6LEI}$$

Values:

P = 150 kN, L = 10 m, a = 6 m, b = 4 m

EXAMPLE 4

Solutions

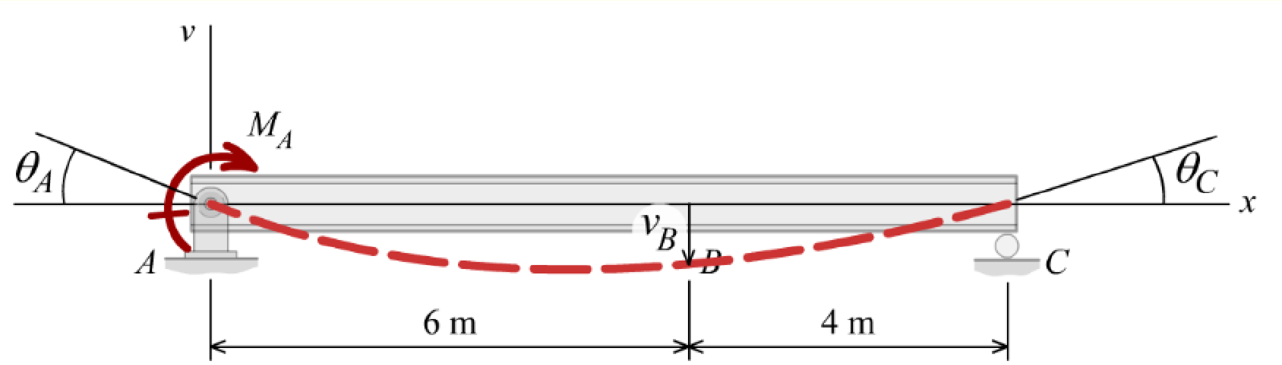
$$\theta_A = -\frac{Pb(L^2 - b^2)}{6LEI} = -\frac{(150\text{ kN})(4\text{ m})}{6(10\text{ m})EI}\left[(10\text{ m})^2 - (4\text{ m})^2\right] = \boxed{-\frac{840\text{ kN-m}^2}{EI}}$$

$$\theta_C = \frac{Pa(L^2 - a^2)}{6LEI} = \frac{(150\text{ kN})(6\text{ m})}{6(10\text{ m})EI}\left[(10\text{ m})^2 - (6\text{ m})^2\right] = \boxed{\frac{960\text{ kN-m}^2}{EI}}$$

EXAMPLE 4

Solutions

- Determine the slopes at A and C caused by moment reaction M_A.



$$\theta_A = -\frac{ML}{3EI} \qquad \text{and} \qquad \theta_C = \frac{ML}{6EI}$$

Values:

$M = M_A$, $L = 10$ m

Calculation:

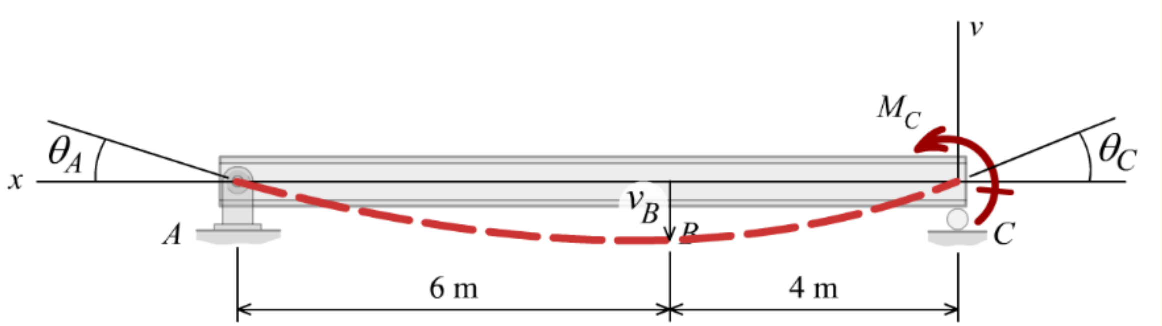
$$\theta_A = -\frac{ML}{3EI} = -\frac{M_A(10\text{ m})}{3EI} = \boxed{-\frac{(3.333333\text{ m})M_A}{EI}}$$

$$\theta_C = \frac{ML}{6EI} = \frac{M_A(10\text{ m})}{6EI} = \boxed{\frac{(1.666667\text{ m})M_A}{EI}}$$

EXAMPLE 4

Solutions

- Determine the slopes at A and C caused by moment reaction M_C..



$$\theta_A = -\frac{ML}{6EI} \quad \text{and} \quad \theta_C = \frac{ML}{3EI}$$

Values:

$M = M_C$, $L = 10$ m

Calculation:

$$\theta_A = -\frac{ML}{6EI} = -\frac{M_C(10\text{ m})}{6EI} = \boxed{-\frac{(1.666667\text{ m})M_C}{EI}}$$

$$\theta_C = \frac{ML}{3EI} = \frac{M_C(10\text{ m})}{3EI} = \boxed{\frac{(3.333333\text{ m})M_C}{EI}}$$

EXAMPLE 4

Solutions

Compatibility equation for slope at *A*:

$$-\frac{840\ \text{kN-m}^2}{EI} - \frac{(3.333333\ \text{m})M_A}{EI} - \frac{(1.666667\ \text{m})M_C}{EI} = 0 \qquad \text{(a)}$$

Compatibility equation for slope at *C*:

$$\frac{960\ \text{kN-m}^2}{EI} + \frac{(1.666667\ \text{m})M_A}{EI} + \frac{(3.333333\ \text{m})M_C}{EI} = 0 \qquad \text{(b)}$$

Solve Equations (a) and (b). Equations (a) and (b) can be rewritten as:

$$(3.333333\ \text{m})M_A + (1.666667\ \text{m})M_C = -840\ \text{kN-m}^2$$

$$(1.666667\ \text{m})M_A + (3.333333\ \text{m})M_C = -960\ \text{kN-m}^2$$

and solved simultaneously for M_A and M_C:

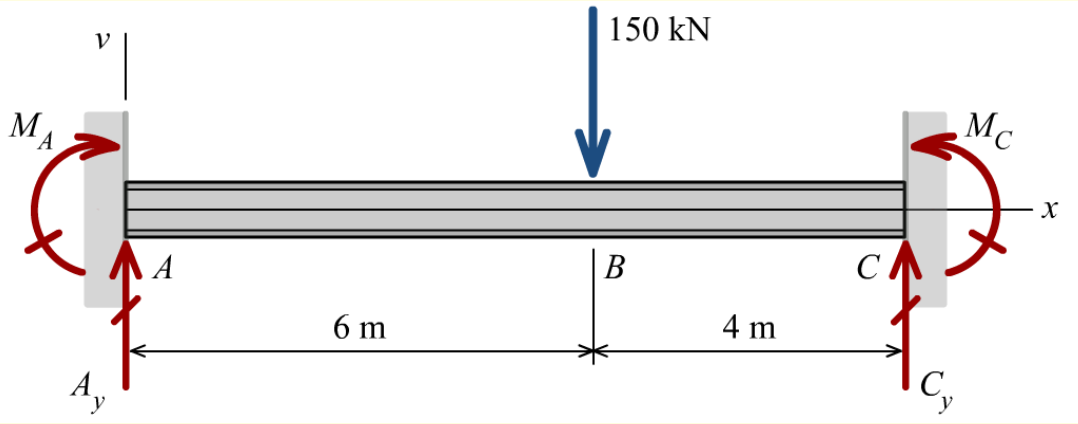
$$M_A = -144\ \text{kN-m} = \boxed{144\ \text{kN-m (ccw)}}$$

$$M_C = -216\ \text{kN-m} = \boxed{216\ \text{kN-m (cw)}}$$

Ans.

EXAMPLE 4

Solutions



$$\Sigma M_A = -M_A + M_C - (150\text{ kN})(6\text{ m}) + C_y(10\text{ m}) = 0$$

$$\therefore C_y = \frac{(150\text{ kN})(6\text{ m}) + (-144\text{ kN-m}) - (-216\text{ kN-m})}{10\text{ m}}$$

$$= 97.2\text{ kN} = \boxed{97.2\text{ kN}\uparrow}$$ **Ans.**

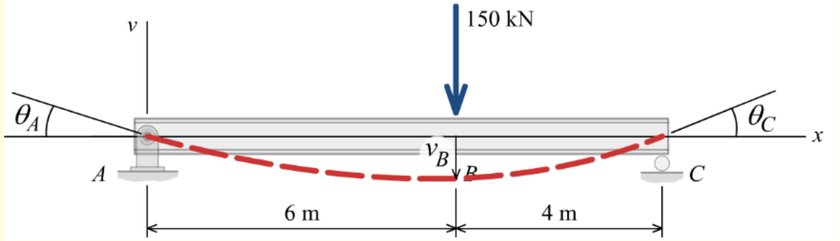
$$\Sigma F_y = A_y + C_y - 150\text{ kN} = 0$$

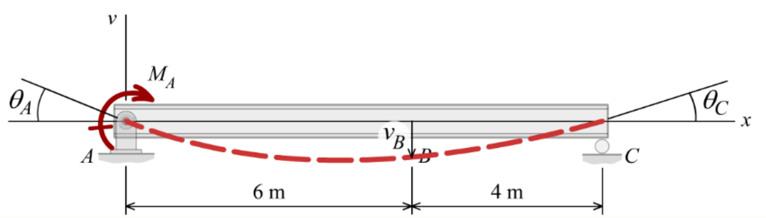
$$\therefore A_y = 150\text{ kN} - 97.2\text{ kN} = 52.8\text{ kN} = \boxed{52.8\text{ kN}\uparrow}$$ **Ans.**

EXAMPLE 4

Solutions

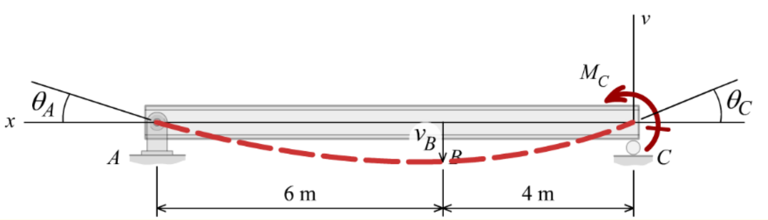
- Beam deflection at B:..





$$= \boxed{\frac{806.4\text{ kN-m}^3}{EI}}$$

$$= \boxed{-\frac{2{,}880\text{ kN-m}^3}{EI}}$$



$$= \boxed{\frac{1{,}382.4\text{ kN-m}^3}{EI}}$$

Beam deflection v_B:

$$v_B = -\frac{2{,}880\text{ kN-m}^3}{EI} + \frac{806.4\text{ kN-m}^3}{EI} + \frac{1{,}382.4\text{ kN-m}^3}{EI}$$

$$= -\frac{691.2\text{ kN-m}^3}{EI} = -\frac{691.2\text{ kN-m}^3}{110{,}800\text{ kN-m}^2} = -0.006238\text{ m} = \boxed{6.24\text{ mm}\downarrow}$$